TRANSACTIONS

OF THE

AMERICAN MATHEMATICAL SOCIETY

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VOLUME 28

1926

PUBLISHED BY THE SOCIETY

MENASHA, WIS., AND NEW YORK

1926

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INTERSECTIONS AND TRANSFORMATIONS OF COMPLEXES AND MANIFOLDS*

BY

SOLOMON LEFSCHETZ

INTRODUCTION

In writing this paper my first objective has been to prove certain formulas on fixed points and coincidences of continuous transformations of manifolds. To this proof for orientable manifolds without boundary is devoted most of the second part, the remainder of which is taken up by a study of product complexes in the sense of E. Steinitz, as they are the foundation on which the proof rests. With suitable restrictions the formulas derived are susceptible of extension to a wider range of manifolds, but this will be reserved for a later occasion. It may be stated that our formulas include and completely generalize the early results due to Brouwer and whatever has been obtained since along the same line.† No such generality would have been possible without that powerful instrument, the product complex.

The principle of the method is best explained by means of a very simple example. Let $f(x)$ and $\varphi(x)$ be continuous and uni-valued functions over the interval 0, 1, and let their values on the interval also lie between 0 and 1. It is required to find the number of solutions of $f(x)=\varphi(x)$, $0\leqq x\leqq 1$.

Graphically the problem is solved by plotting the curvilinear arcs

$$y=f(x), \quad y=\varphi(x), \quad 0\leqq x\leqq 1,$$

and taking their intersections. A slight modification of the functions may change the number of solutions, even make them become infinite in number. However, the difference between the numbers of *positive* and *negative* crossings of sufficiently close polygonal approximations to the arcs is a fixed number, their Kronecker index. Its determination is then a partial answer to the question, and indeed seemingly the only possible general answer.

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* Presented to the Society under somewhat different title at the Chicago Meeting of April 13, 1923, and the Southwestern Section Meeting of December 1, 1923; received by the editors in November, 1924.

† A good bibliography is found in Kérékjárto's recently published volume *Topologie*, Berlin, J. Springer. For a list of the most recent titles see a paper by J. W. Alexander, these Transactions, vol. 25 (1923), p. 173, to which must be added my notes in the Proceedings of the National Academy of Sciences, vol. 9 (1923), p. 90, vol. 11 (1925), pp. 287, 290, summarizing the results of the present paper.

The two complexes whose product is taken in this case are the unit segments on the x and y axes, their product being the square whose sides they are. Replace the unit segments by two identical manifolds of n dimensions, M_n and M_n', the square by the M_{2n} image of their pairs of points (product of the two), the arcs by manifolds on M_{2n} and the exact situation of Part II is obtained.*

In all questions of the above type, the Kronecker index plays then an essential part. In order to put everything on a solid basis, it seemed vital to discuss thoroughly this index. There is in existence an excellent treatment of it by Hadamard,† leaving little to be desired for euclidean spaces, but distinctly insufficient for general manifolds. Then the Kronecker index is only a special topic in the more interesting and far reaching theory of the intersection of complexes on a manifold,‡ needed in any case, to some slight extent, for a good treatment of the index itself. To this theory is devoted most of Part I, of which the chief result is as follows: given several complexes on an orientable manifold M_n, which do not intersect on each other's boundaries nor on that of M_n, there exists a well defined cycle of M_n, their *intersection*. It is well defined in this sense: no matter how the complexes are approximated by means of straight complexes, the cycle intersection of the latter remains homologous to itself. If the approximating complexes intersect in isolated points there is a definite Kronecker index independent of the mode of approximation.

The independence from covering complexes and related modes of defining straightness has presented some of our most serious difficulties. It is a little surprising that the necessity of freeing the Kronecker index from this vitiating circumstance has never been considered in the literature. That the wider problem has not been attacked is natural enough since intersections of general complexes have been studied but very little if at all.§

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* This concept appeared first, applied to the special case of algebraic correspondences, in Severi's paper in the Torino Memorie, vol. 54 (1904). Needless to say, Severi did not suspect the analysis situs aspect of the problem, hence did not and could not derive the Hurwitz coincidence formulas. See in this connection Enriques and Chisini, *Lezioni sulla Teoria Geometrica delle Equazioni*, vol. 3, p. 427, also Chisini, Istituto Lombardo Rendiconti, ser. 2, vol. 7 (1924), p. 481. Their work is anticipated by my first Note, which seems to have escaped their notice.

† Note to Tannery's *Introduction à la Théorie des Fonctions*. See also the first chapter of my recent Borel Series Monograph, *L'Analysis Situs et la Géométrie Algébrique*, and my paper in the 1921 Transactions which both contain important applications of the index to algebraic geometry, and finally a very interesting paper by Veblen that has just appeared in these Transactions, vol. 25 (1923); results are recalled in Part I, §7 of this paper and derived anew in Part II.

‡ Considered and actually applied, I believe, for the first time in my Monograph.

§ Some very important results along that line have been obtained of late by J. W. Alexander. See Proceedings of the National Academy of Sciences, vol. 10 (1924), pp. 99, 101, 493.

PART I. QUESTIONS OF INTERSECTION

§1. PRELIMINARIES

1. In notation and terminology, we shall follow essentially Veblen's *Colloquium Lectures on Analysis Situs*. We shall assume the reader fairly familiar with this fundamental work and briefly refer to it as "Coll. Lect." The following designations will recur with special frequency in our paper: S_n = euclidean n-space; $E_n = n$-cell; $C_n = n$-dimensional complex; $M_n =$ manifold of n dimensions; Γ_n (also γ_n, δ_n in Part II) $= n$-cycle.* The various numerical invariants, the signs $\sim$, $\equiv$, for congruence or homology, and also the definition of orientation are as in Chapter I of my Borel Series Monograph, *L'Analysis Situs et la Géométrie Algébrique*. In Part II we shall introduce the sign $\approx$ for homologies with division allowed, that is with zero-divisors neglected.

2. Our ordinary complexes shall be restricted to Veblen's *regular* type. Such a C_n is the homeomorph of an n-dimensional polyhedron Π_n whose faces are all *simplicial* cells (interiors of simplexes) no two intersecting. The cells of Π_n define those of C_n. We shall apply the term rectilinear segment, polygonal or polyhedral configuration, etc., to C_n, as if it were Π_n itself, meaning thereby the images of the Π_n configurations. Distances on C_n shall also be measured by reference to Π_n, which for the purpose is assumed immersed in some $S_{n'}$, $n' \leqq n$. By a *subcomplex* of C_n we shall mean one made up with cells of C_n.

C_n may be subdivided into new complexes, and this can be carried out so that the cells of the new complex be of diameter $< \epsilon$, assigned. Of importance in this connection is the method of *regular* subdivision (Coll. Lect., p. 89).

3. We shall define manifolds in accordance with a suggestion due to Veblen (Coll. Lect., p. 92). It amounts essentially to demanding of a C_n defining an M_n that its cells be grouped about any particular one much as if they were all immersed in an S_n. It will be found worth while to examine the matter at closer range.

Let E_k, $k < n$, be any simplicial cell on C_n, E_{k+1} another incident with it (i. e. with E_k on its boundary). We define a set $\{e\}$ of elements such that (*a*) to every E_{k+1} corresponds one and only one e; (*b*) to e corresponds together with a given E_{k+1} all others having such a cell in common with it; (*c*) two elements of the set are said to tend towards one another if and

* Or set of oriented n-circuits in Veblen's terminology.

only if there are corresponding cells whose vertices opposite to E_k tend towards one another. This last condition gives a definition of continuity for $\{e\}$.

Now C_n is said to define an M_n *without boundary* if for every E_k, $0 \leqq k \leqq n-1$, the set $\{e\}$ is homeomorphic to the boundary of an $(n-k)$-cell. The M_n *with boundary* is then defined as in Coll. Lect., p. 88.

To verify the manifold condition we proceed much as in Coll. Lect., pp. 88, 90. We shall show in §3 (No. 14, Lemma II) that there exists a subdivision C_n' of C_n of which $E_k = A_0A_1 \cdots A_k$ is a cell, the A's being its vertices. (Incidentally this method, now quite customary, of naming a cell by its vertices will prove very convenient.) Let $E_h = A_0A_1 \cdots A_h$ be any cell of C_n' incident with E_k. Then $A_{k+1} \cdots A_h$ is also a cell of C_n', and the totality of such cells gives rise to a C_{n-k-1} homeomorphic to $\{e\}$. Hence the manifold condition is equivalent to demanding that all these complexes be homeomorphic to cell boundaries. As the complexes for a given E_k are all homeomorphic the verification is really independent of the particular C_n' chosen.

It is as yet unknown whether for a given M_n the manifold condition is verified simultaneously for all defining complexes. We shall therefore agree to consider only defining complexes such that the cells of M_n for which the condition is verified have for logical sum one and the same point set.

4. The *orientation* of a simplicial cell is best defined by the order of naming the vertices (Monograph, p. 13). The oriented C_n is a complex as previously understood plus an assigned orientation for each n-cell. With the orientation of E_n there is attached one for its boundary $(n-1)$-cells. M_n is called *orientable* if its defining C_n can be so oriented that every non-bounding E_{n-1} receives opposite orientations from its two adjacent E_n's. This property is independent of the particular defining C_n. For match the complex with a copy of itself so that corresponding boundary points coincide. There will result a set of n-circuits, orientable or not at the same time as C_n itself. From the known independence for the circuit (Coll. Lect., pp. 100-102) follows that for M_n.

5. It is frequently convenient to orient M_n by means of a special n-cell used as indicatrix, thus: on an $E_n = A_0A_1 \cdots A_n$ of the defining complex we choose another $E_n' = B_0B_1 \cdots B_n$ reducible to the first by an affine transformation of the common S_n with coincidence of vertices in the order named, and instead of assigning the order of the A's, we do it for the B's.

Let $x_1, x_2, \cdots, x_n$ be cartesian coördinates on S_n, the origin being B_0. Then E_n' is completely defined if we give ourselves the matrix H' of the coördinates of the B's, it being understood that the ith row corresponds

to B_i. Let E_n'' be another indicatrix with the same first vertex B_0, and let H'' be the corresponding matrix. Then E_n'' is the indicatrix of $+M_n$ or $-M_n$ according as the signs of the determinants of H', H'' (certainly $\neq 0$) are or are not the same.

Remark. Whenever we derive from C_n a new complex C_n', subdivision of the first, we shall agree to orient each n-cell of C_n' so that it constitutes an indicatrix of the n-cell of C_n that carries it.

6. With Veblen we shall call *singular* k-cell on C_n a point set $\bar{E}_k$, of C_n, uniform and continuous image of an ordinary cell E_k which we may as well assume simplicial. Any statement concerning $\bar{E}_k$, in particular regarding its orientation or boundary cells, is to be interpreted by reference to E_k.

Let $E_k^1, \cdots, E_k^p$ be sensed cells on C_n, and let $E_{k-1}^1, \cdots, E_{k-1}^q$ be their bounding $(k-1)$-cells, all cells being possibly (but not necessarily) singular. We shall extend the term "k-complex on C_n" to cover a symbol

$$C_k = \sum x_i E_k^i$$

where the x's are arbitrary integers. The points of C_k are those of the E_k's whose x coefficient is not zero plus their limit points. To the cells correspond Poincaré congruences

$$E_k^i \equiv \sum y_{ij} E_{k-1}^j ,$$

and for C_k by definition

$$C_k \equiv \sum x_i y_{ij} E_{k-1}^j .$$

The C_{k-1} at the right is the *boundary* of C_k. If it reduces to zero, C_k is a *k-cycle*. The fact that C_{k-1} is a boundary is also expressed by the homology

$$C_{k-1} \sim 0 \pmod{C_n} .$$

Remark. The boundary (sensed) of an n-cell is a cycle; the verification is immediate. Hence, by summation the boundary of a C_n is also a cycle.

§2. Intersection of cells

7. Let E_h, E_k, E_n be simplicial cells, the first two on the third. E_h and E_k may intersect in various ways. Assume that in the S_n of E_n the spaces S_h and S_k which carry the other cells are linearly independent, so that their intersection is an S_l, $l = h+k-n$. Grant furthermore that $l \geqq 0$, so that S_l is an actual space (possibly a point) and also that the cells themselves intersect. This intersection will consist of an element of S_l, bounded by a convex polyhedron, and therefore constitutes an l-cell E_l. If we assume the linear spaces of the boundaries of E_h and E_k also as independent as possible

from those of the cells themselves, no bounding $(l-1)$-cell of E_l (necessarily polyhedral like E_l itself) will be an $(l-1)$-cell of the boundary of E_h or E_k, but will be merely on one of their bounding $(h-1)$-cells or $(k-1)$-cells respectively, and this we shall assume for the present. E_l with its bounding polyhedron constitutes an M_l decomposable into simplexes, for example by the regular subdivision process. To this M_l we now propose to assign an orientation corresponding to given orientations of the other cells as follows. Let $E_n' = A_0A_1 \cdots A_n$ be a small simplicial cell such that $E_l' = A_0A_1 \cdots A_l$ lies on E_l, $E_h' = A_0A_1 \cdots A_h$ on E_h, and $E_k' = A_0A_1 \cdots A_l A_{h+1} \cdots A_n$ on E_k. Let a_sE_s', $a_s = \pm 1$, be the indicatrix of E_s $(s = h, k, l, n)$. Then a_l is to be determined by the relation

$$a_h \cdot a_k \cdot a_l \cdot a_n = +1 .$$

The cell so sensed shall be designated by $E_h \cdot E_k$.*

The relation between the a's shows that if one of the cells E_h, E_k, E_n is inverted so is E_l. Furthermore if E_h and E_k are permuted the only indicatrix changed is E_n', whose vertices undergo $(n-h)(h-l) = (n-h)(n-k)$ transpositions. Hence

$$E_h \cdot E_k = (-1)^{(n-h)\cdot(n-k)} E_k \cdot E_h = -(-E_h) \cdot E_k = -E_h \cdot (-E_k) .$$

8. We have tacitly assumed throughout that $l > 0$. With suitable conventions we may let it take any value whatever. The case $l < 0$ may be dismissed at once; we simply write, then,

$$E_h \cdot E_k = 0.$$

Let now $l = 0$. Then there is a unique point of intersection constituting a zero-cell E_0. It is this point with the value of a_0 attached which we designate by $E_h \cdot E_{n-h}$. The value of a_0 is called the *Kronecker index* or simply index of E_h and E_{n-h} and denoted by $(E_h \cdot E_{n-h})$. The above symbolic relation still holds, and we derive from it and our discussion

$$(E_h \cdot E_{n-h}) = (-1)^{(n+1)h} (E_{n-h} \cdot E_h) = -(-E_h \cdot E_{n-h}) = -(E_h \cdot -E_{n-h}) ,$$

which may also be obtained directly by means of the indicatrices.

To make our conventions complete, when the cells do not intersect, we shall write

$$E_h \cdot E_k = 0 , \qquad (E_h \cdot E_{n-h}) = 0 .$$

The case $h = 0$, $k = n$, is not exceptional. We then have a point $E_0 = A$ and an attached unit a_0 in place of a_h. The point is on the "intersecting"

* In previous papers the same notation was used, sometimes with, sometimes without the "dot." In this paper it has been essential to use the dot throughout, for in Part II another "product" symbol comes in, whose meaning is wholly different and which will be written as a "cross" product.

cell $\bar{E}_n$ (in place of E_k) and a_k is replaced by $\bar{a}_n$ whose value is $+1$ if E_n is sensed like E_n, -1 otherwise. The index is now

$$(E_0 \cdot E_n) = a_0 = a_n \cdot \bar{a}_n \cdot \bar{a}_0 .$$

Its sign is that of $\bar{a}_0$ if E_n and $\bar{E}_n$ are sensed alike, its opposite otherwise.

9. Our next task is to determine the boundary congruences. We first assume that $E_h \cdot E_k$ *has no boundary* $(l-1)$*-cell on the boundary of* E_n. This is indeed the general case, but the exception here referred to is of importance later. Let the boundary congruences for E_h and E_k be

$$E_h \equiv \sum E_{h-1}^i , \qquad E_k \equiv \sum E_{k-1}^j .$$

The boundary of $E_h \cdot E_k$ is then the sum of the cells $E_h \cdot E_{k-1}^j$, $E_{h-1}^i \cdot E_k$, affected with signs that are to be determined.

Let for example E_h actually intersect E_{k-1}^j and choose E_n' with the vertices of

$$E_{k-1}' = A_0 A_1 \cdot \cdot \cdot A_{l-1} \, A_{h+1} \cdot \cdot \cdot A_n \text{ on } E_{k-1}^j .$$

As A_l must be transposed l times to come to first place, $(-1)^l \cdot a_k \cdot E_{k-1}'$ is an indicatrix of E_{k-1}^j. Hence, a_{k-1} corresponding to E_{k-1}^j as a_k to E_k, we have

$$a_{k-1} = (-1)^l a_k .$$

Therefore

$$a_{l-1} = (-1)^l \cdot a_l$$

corresponds to $E_h \cdot E_{k-1}$ as a_l to $E_h \cdot E_k$ itself, so that $E_h \cdot E_{k-1}$ has for indicatrix $(-1)^l \cdot a_l \cdot A_0 A_1 \cdot \cdot \cdot A_{l-1}$, which is the precise indicatrix that it should have as a boundary cell of $E_h \cdot E_k$, since A_l must be transposed l times to be brought to first place, and since $a_l A_0 A_1 \cdot \cdot \cdot A_l$ is the indicatrix of $E_h \cdot E_k$.

We conclude then that in the boundary congruence for $E_h \cdot E_k$ we must affect $E_h \cdot E_{k-1}$ with the sign $+$. Similarly $E_k \cdot E_{h-1}$ must be affected with the sign $+$ in the congruence for $E_k \cdot E_h$, hence $(-1)^{(n-h+1)(n-k)} E_{h-1}^i \cdot E_k$ with the sign $+$ in the congruence for $(-1)^{(n-h)(n-k)} E_h \cdot E_k$, from which at once

$$E_h \cdot E_k \equiv (-1)^{(n-k)} \sum E_{h-1}^i \cdot E_k + \sum E_h \cdot E_{k-1}^j .$$

In the exceptional case at first excluded, E_l will have some bounding E_{l-1}'s on some bounding E_{n-1} of E_n. This will be due to the fact that for example E_{h-1}^p and E_{k-1}^q will both lie in E_{n-1}. It is found by considering now

intersections in E_{n-1} that $E^p_{h-1} \cdot E^q_{k-1}$ is positively related to $E_h \cdot E_k$. The method is the same as above: we merely assume in our indicatrix A_l exterior to E_{n-1} and the rest goes through about as before. We shall then have the general congruence

$$(9.1) \quad E_h \cdot E_k \equiv (-1)^{n-k} \sum E^i_{h-1} \cdot E_k + \sum E_h \cdot E^j_{k-1} + \sum E^p_{h-1} \cdot E^q_{k-1} .$$

We have here a case where it would be distinctly worth while to have a more complicated notation to indicate in which complex intersections are taken. Such instances are comparatively rare and the doubt will always readily be cleared up by reference to the context.

10. **Fundamental theorem on Kronecker indices.** Just as before, the case where $l=1$, and the cells at the right are points (zero-cells) offers no exception. $E_h \cdot E_k$ is then a rectilinear one-cell E_1 and the three sums at the right reduce to two terms corresponding to the initial and terminal points of E_1, the sensed intersection. To each of these terms corresponds a Kronecker index, computed either as to E_n or as to one of its bounding $(n-1)$-cells. I say, and this is our theorem, that *in all cases the sum of these two-indices is zero* so that they are units of opposite signs.

11. Let then

$$E_1 = E_h \cdot E_{n-h+1} \equiv (-1)^{h-1} E'_{h-1} \cdot E_{n-h+1} + \cdots ,$$

where we have not written the term that we do not wish to discuss. Assume first that the term written represents the initial point A_0 of E_1 and take it for vertex of same name of the indicatrix previously considered. Here then $a_l = a_1 = +1$. The situation is as follows:

$$\begin{array}{rcl} A_0A_2 \cdots A_h & \text{is indicatrix for} & -a_h \cdot E'_{h-1} , \\ A_0A_1A_{h+1} \cdots A_n & " \quad " \quad " & a_{n-h+1} \cdot E_{n-h+1} , \\ A_0A_2 \cdots A_hA_1A_{h+1} \cdots A_n & " \quad " \quad " & (-1)^{h-1} \cdot a_n E_n . \end{array}$$

The Kronecker index for the point $(-1)^{h-1} \cdot E'_{h-1} \cdot E_{n-h+1}$ is then the number β defined by the condition

$$(-1)^h \cdot a_h \cdot a_{n-h+1} \cdot (-1)^{h-1} \cdot a_n \cdot \beta = 1 .$$

We have also

$$a_h \cdot a_{n-h+1} \cdot a_n \cdot a_1 = a_h \cdot a_{n-h+1} \cdot a_n = 1 .$$

Hence finally $\beta = -1$. If A_0 were the terminal point of E_1, we would have merely $a_1 = -1$, the rest being the same, hence $\beta = +1$. Thus we see that the Kronecker indices for the end points have the same signs that the points receive in the boundary congruence for E_1. The other two cases (where A_0 is $E_h \cdot E_{n-h}$ or $E_{h-1} \cdot E_{n-h}$ and on an E_{n-1}) lead to exactly the same con-

clusion; the proofs, essentially similar, are omitted here. The sum of the indices is therefore always zero as was to be proved.

12. The extension to the intersection of s cells E_i, E_h, $\cdots$, E_k on E_n, goes through with ease. It is denoted by $E_i \cdot E_h \cdots E_k$, a symbol which obeys the associative but in general not the commutative law. A similar remark holds for the index $(E_i \cdot E_h \cdots E_k)$, which exists only when $i+h+\cdots+k=n(s-1)$. The boundary congruences can be written down at once.

§3. Intersections of polyhedral complexes and their Kronecker indices

13. The complexes which are to occupy us in the rest of Part I shall all be immersed in a connected, orientable and oriented manifold M_n with an assigned defining complex C_n. We shall assume throughout that intersecting complexes have non-intersecting boundaries and no common points on the boundary C_{n-1} of M_n. Of several intersecting complexes so restricted let one, say C_h, have points on C_{n-1}. We may subdivide C_h into C_h' with cells so small that those h-cells which have points of C_{n-1}, or whose boundary has some, carry no points of the intersecting complexes on themselves or on their boundary. Let $\bar{C}_h$ be the complex sum of these cells plus their boundaries. As far as the intersection with the other complexes is concerned, C_h' may be replaced by $C_h'-\bar{C}_h$ which carries no points of C_{n-1}. A similar remark applies in case there are points common to some, but not all, the boundaries of the intersecting complexes. Henceforth it shall then be understood once for all that

I. *intersecting complexes have no points on the boundary of* M_n;

II. *their boundaries do not meet.*

Our general plan is as follows. We shall first define the intersection of a still narrower class of polyhedral complexes, and then approximate general complexes by means of these. But before defining our special polyhedral complexes, we must prove two lemmas.

14. Lemma I. *Any polyhedral* C_h *is a sum of simplicial cells.*

Each h-cell of C_h is a sum of a finite number of polyhedral regions of a certain S_h. Each region is decomposable into a sum of convex polyhedral h-cells.* Remove these from C_h and let C_{h-1} be the remaining complex. The lemma is true for $h=1$. Grant it for the dimensionality $h-1$; C_{h-1} can then be decomposed into a sum of simplicial cells. Select a point on each convex h-cell and join it by rectilinear segments to the simplicial

* This has been proved by Veblen and others. For detailed references see Coll. Lect., p. 83.

cells of C_{h-1} on the boundary of its h-cell. There will follow the requisite decomposition of C_h.

Remark. A region of the initial decomposition of C_h may lie on several cells of C_n. If so the boundaries of the latter decompose it into regions each of which lies on a unique cell. Then C_h will appear as a sum of simplicial cells, each also on a unique cell of C_n. Whenever we shall consider in the sequel a polyhedral C_h, arising in some manner in the course of the discussion, we shall assume that it *has been decomposed into a sum of simplicial cells each on a unique cell of* C_n. Strictly speaking, the initial complex is thus replaced by a subdivision and should be designated by a new notation, but it will simplify matters a good deal to avoid this.

LEMMA II. *There exists a subdivision C_n' of C_n with C_h as a sub-complex.*

Decompose C_h as just stated into a sum of simplicial cells, any one, say E_k, on an E_n of C_n or on its boundary. The S_k of E_k is the intersection of certain S_{n-1}'s of the S_n of E_n. Extend these S_{n-1}'s as far as possible on the simplex of their E_n. There will result a decomposition of C_n into a new complex $\bar{C}_n$ with C_h as a subcomplex. Apply now Lemma I to $\bar{C}_n$ and C_n' follows.

15. We now seek to define the intersection of two polyhedral complexes C_h, C_k and its boundary congruences. Let

$$C_h = \sum E_h^i ; \qquad C_k = \sum E_k^j ;$$

$$E_h^i \equiv \sum E_{h-1}^{ip} ; \qquad E_k^j \equiv \sum E_{k-1}^{jq} .$$

We impose the following restrictive conditions:

(*a*) *Intersecting h- and k-cells are on one and the same n-cell of C_n and in general position as understood in* §2. *Their intersection is then an l-cell, where, as before, $l=h+k-n$.*

(*b*) *Let E_{h-1}^{ip} intersect E_{k-1}^{jq} on an E_{l-1}. Then both are on an E_{n-1} of C_n and E_{l-1} is not on the boundaries of C_h and C_k.*

When these two conditions are satisfied, the intersection, to be denoted by $C_h \cdot C_k$, is a C_l defined by the relation

$$C_h \cdot C_k = \sum E_h^i \cdot E_k^j ,$$

it being understood that, whenever E_h^i and E_k^j do not intersect, $E_h^i \cdot E_k^j = 0$.

The symbols $C_h \cdot C_k$ obeys the distributive law, as follows at once from the definition. Thus:

$$(C_h' + C_h'') \cdot C_k = C_h' \cdot C_k + C_h'' \cdot C_k ,$$

it being granted of course that both C_h' and C_h'' satisfy conditions (a) and (b) as to C_k. Similarly for C_k while the effect of permuting the C''s or inverting them is as in No. 7 for the cells.

The boundary congruences will present no great difficulty. From (9.1) follows

$$(15.1) \qquad C_h \cdot C_k \equiv (-1)^{n-k} \sum E_{h-1}^{ip} \cdot E_k^j + \sum E_h^i \cdot E_{k-1}^{jq} + \sum E_{h-1}^{ir} \cdot E_{k-1}^{js},$$

the meaning of each sum being readily apprehended by reference to (9.1). I say that the terms in the third sum cancel each other. Indeed let E_h^i and E_k^j be on the cell E_n and give rise to the term $E_{h-1}^{ir} \cdot E_{k-1}^{js}$, intersection of E_{h-1}^{ir}, E_{k-1}^{js}, situated in a bounding cell E_{n-1} of E_n. We assume the cells E_{h-1}^{ir}, E_{k-1}^{js}, E_{n-1} positively related to E_h^i, E_k^j, E_n, and E_{h-1}^{ir}, E_{k-1}^{js} is the intersection of the two sensed cells, oriented as indicated in §2.

According to (b) there exist E_n', $E_h^{i'}$, $E_k^{j'}$ of C_n, C_h, C_k, with E_{n-1}, E_{h-1}^{ir}, E_{k-1}^{js} on their boundaries and negatively related to them. There is a cell labelled $E_{h-1}^{i'r'} = -E_{h-1}^{ir}$ and one labelled $E_{k-1}^{j's'} = -E_{k-1}^{js}$. Indeed according to ($b$) the cells of C_h adjacent to E_{h-1}^{ir} can be grouped in pairs oppositely related, and we may assume that we have such a pair in E_h^i, $E_h^{i'}$. Similarly for E_k^j and $E_k^{j'}$.

There are now several possibilities. It may be that E_{n-1} separates E_h^i and $E_h^{i'}$ (that is, one of them is on E_n, the other on E_n') but not E_k^j and $E_k^{j'}$. Then the third sum in (15.1) contains these terms pertaining to the couples considered above and no others:

$$E_{h-1}^{ir} \cdot E_{k-1}^{js} + E_{h-1}^{ir} \cdot E_{k-1}^{j's'}.$$

They represent intersections on E_{n-1} and as $E_{k-1}^{j's'} = -E_{k-1}^{js}$ their sum is zero.

A second possibility is that E_{n-1} separates the two pairs of h- and k-cells. Then in the sum in question there correspond the terms

$$E_{h-1}^{ir} \cdot E_{k-1}^{js} + E_{h-1}^{i'r'} \cdot E_{k-1}^{j's'}.$$

The first intersection is taken on E_{n-1}, the second on $-E_{n-1}$ (that is in the scheme of §2, E_{n-1} must now be replaced by $-E_{n-1}$, the reason being that boundaries of cells on E_n' are now involved and E_n' is negatively related to E_{n-1}). When intersections are referred to E_{n-1}, the second term must be written

$$-E_{h-1}^{i'r'} \cdot E_{k-1}^{j's'} = -(-E_{h-1}^{ir}) \cdot (-E_{k-1}^{js}) = -E_{h-1}^{ir} \cdot E_{k-1}^{js}$$

and the sum is again zero.

Finally we must consider the case where E_{n-1} separates none of the two pairs of cells. Then the terms to be considered are now

$$E^{ir}_{k-1} \cdot E^{js}_{k-1} + E^{ir}_{k-1} \cdot E^{j's'}_{k-1} + E^{i'r'}_{k-1} \cdot E^{js}_{k-1} + E^{i'r'}_{k-1} \cdot E^{j's'}_{k-1} ,$$

the intersections being all referred to E_{n-1}. It is immediately verified that the fourth term is the same as the first, and the other two terms its negative. The sum is then again zero. This completes the proof of our assertion.

16. The boundary of C_h is a cycle Γ_{h-1}, and that of C_k is a cycle Γ_{k-1}. We have

$$C_h \equiv \Gamma_{h-1}, \quad C_k \equiv \Gamma_{k-1} .$$

The first two sums in (15.1) are respectively $\Gamma_{h-1} \cdot C_k$ and $C_h \cdot \Gamma_{k-1}$. Hence in the last analysis we have this fundamental congruence:

$$(16.1) \qquad C_h \cdot C_k \equiv (-1)^{n-k} \Gamma_{h-1} \cdot C_k + C_h \cdot \Gamma_{k-1} .$$

17. A series of important corollaries follows at once from the preceding discussion.

I. A complex C_h is called a *generalized manifold* if every non-bounding E_{h-1} of it is incident with just two h-cells. Orientability is defined for it as for an ordinary manifold. From our discussion we obtain the following: *if C_h and C_k are orientable generalized manifolds so is $C_h \cdot C_k$.*

II. *If one of the complexes is a cycle the boundary of $C_h \cdot C_k$ is the intersection of this cycle with the other complex or its opposite. If both are cycles so is their intersection.* In symbols,

$$(17.1) \qquad C_h \cdot \Gamma_k = (-1)^{n-k} \Gamma_{h-1} \cdot \Gamma_k ; \quad \Gamma_h \cdot C_k \equiv \Gamma_h \cdot \Gamma_{k-1} ; \quad \Gamma_h \cdot \Gamma_k \equiv 0 .$$

III. *When the boundary of each complex does not meet the other, $C_h \cdot C_k$ is a cycle.*

IV. *Let Γ_h bound C_{h+1} satisfying our restrictive conditions as to its intersection with C_k. Then $\Gamma_h \cdot C_k$ is a bounding cycle also.*

This can be read off from (17.1).

18. **Kronecker index.** This time the dimensions are h and $n-h$. We make the same assumptions as previously, and, in addition, agree that for two non-intersecting cells the index is zero. Then

$$(C_h \cdot C_{n-h}) = \sum (E^i_h \cdot E^j_{n-h}) .$$

From No. 8 follows the distributive law for indices. The result of permuting the two complexes or of changing the sign of one is as for cells and need not be written down.

From No. 10 and (16.1) it follows that if

$$C_h \equiv \Gamma_{h-1}\,, \quad C_{n-h+1} \equiv \Gamma_{n-h}\,,$$

then here

$$(C_h \cdot \Gamma_{n-h}) = (-1)^h \cdot (\Gamma_{h-1} \cdot C_{n-h+1})\,, \tag{18.1}$$

a formula of great importance later (§6), as it transforms an index corresponding to dimensionalities h, $n-h$, to one with h replaced by $h-1$.*

19. From the theorem of No. 10, together with (16.1), now follows

Let Γ_h bound a C_{h+1} not intersecting the boundary of C_{n-h}, a condition that disappears if we deal with a Γ_{n-h}. Let furthermore the usual restrictions as to intersecting complexes, C_{h+1} and C_{n-h} or Γ_{n-h}, hold. Then

$$(\Gamma_h \cdot C_{n-h}) = 0\,, \quad (\Gamma_h \cdot \Gamma_{n-h}) = 0\,.$$

Observe that owing to the distributive law, it is not necessary that $\Gamma_h \sim 0$, but merely ≈ 0, for then $t\Gamma_h$ bounds and the multiples of the indices are zero; hence also the indices themselves. This result will have important applications in Part II.

20. The extension to several intersecting complexes offers no particular difficulties. The symbols follow the associative and distributive laws, but not in general the commutative law.

§4. Approximation of complexes

21. A first, but somewhat inelastic, approximation to a general complex C_k by a polyhedral C_k' will be obtained by direct application of processes due to Alexander (these Transactions, vol. 16 (1915), p. 148) and Veblen (Coll. Lect., pp. 95, 118). C_k appears then as a subcomplex of a subdivision of C_n with cells of suitably small diameter. There are two associated complexes C_{k+1} and C_k^0 such that

$$C_{k+1} \equiv C_k - C_k' + C_k^0\,, \tag{21.1}$$

and therefore

$$C_k' \sim C_k + C_k^0\,. \tag{21.2}$$

C_k^0 appears only when C_k is not a cycle. Our C_{k+1} is the same as Veblen's B_{i+1}. When C_k is not a cycle, the boundary cells of C_{k+1} which join boundary cells of C_k and C_k' are also part of the boundary of C_{k+1}, and their sum is precisely C_k^0.

* An analogous formula for cells was given by Veblen in the Transactions paper already quoted. p. 542.

In the light of this and upon examining the construction, we find by the simplest continuity considerations that it may be carried out so that the following statements hold.

(*a*) *C_k' and C_{k+1} are both as near as we please to C_k.*

(*b*) *The complex C_k^0 and the boundary of C_k' are as near as we please to the boundary of C_k.*

(*c*) *C_k' includes any particular polyhedral boundary subcomplex of C_k.*

This is proved by a very transparent application of Lemma II, No. 14.

22. From the preceding method of approximation we may derive this interesting result: Let the approximation be made by means of cells of C_n. *If the points of a cell of C_k are sufficiently near those of E_h of C_n, then its approximation is E_h itself or a cell on its boundary. The cell of C_k need only be within a certain distance δ of E_h in order that this be true.*

From this we have the following

THEOREM. *To every polyhedral complex C_k corresponds a positive number δ such that every cycle Γ_h whose points are all within δ of C_k is homologous to a cycle Γ_h' on C_k.*

For C_k is a subcomplex of a subdivision C_n' of C_n (Lemma II, No. 14) and in this case C_h^0 corresponds to C_k^0 of No. 21, for Γ_h is absent. We can also affirm that $\Gamma_h - \Gamma_h'$ will bound, by (*a*), a C_{h+1} whose points are as near as we please to C_k at the same time as those of Γ_h.

Incidentally, since C_k can have no cycle of more than k dimensions, we have this very interesting result: *A non bounding cycle cannot be homologous to a cycle as near as we please to a complex of smaller dimensionality.*

23. The approximations which we have obtained so far are not flexible enough for our purpose, which demands the approximation of two or more complexes at the same time by others with a well-defined intersection. This will be based upon the all-important

THEOREM. *Let C_k be a subcomplex of C_n. By subtraction of bounding k-cycles, it may be reduced to another complex with the same boundary, whose $(k-i)$-cells not on the boundary of M_n nor on its own are on cells of at least $n-i$ dimensions of C_n.*

Let C_n' be a regular subdivision of C_n and C_k' the corresponding subdivision of C_k. Any vertex of C_n' shall be affected with an upper index, such as A_i^p to indicate the dimensionality p of the cell of C_n which carries it.

Let us attach to any cell $E_r = A_0 A_1 \cdots A_r$ of C_n' a symbol $(p_0, p_1, \cdots, p_r)$ to describe its type. Observe that *the p's are all distinct*, for $p_0 = p_1 = p$ would mean that two points on distinct p-cells of C_n are joined by a rectilinear segment wholly on a p-cell.

Another and more significant property is that *the highest p indicates the dimensionality of the cell of C_n which carries E_r.* For then E_r has points on such a cell (in the vicinity of the corresponding A^p); hence, due to the mechanism of regular subdivision, it lies entirely on it.

Our theorem will then be proved if we can show that C_k *is reducible to a complex of which every E_{k-i} not on its boundary nor on that of M_n has in its symbol an integer $\geqq n-i$.*

24. (*a*) For $n=1$, the reduction is immediate. Then $k=0$ alone needs to be considered. C_0 is a sum of points each with an assigned sign. Any such point A, say affected with $+$, may be reduced to any point B of C_1 by adding the end points of a polygonal line AB. These end points constitute the bounding Γ_0 of the theorem.

(*b*) Let now $i=0$ and E_k be as yet not reduced. Its symbol is then of type $A_0^{p_0}\cdots A_k^{p_k}$ with all the p's less than n. The cell lies therefore on the boundary of an n-cell of C_n on which there is a vertex A_{k+1}^n. Let Γ_k be the boundary of the simplex $A_{k+1}^n A_0^{p_0}\cdots A_k^{p_k}$ which is positively related to E_k; $C_k'-\Gamma_k$ is a complex which has the same structure as C_k except that E_k has been replaced by $k+1$ cells of same dimensionality in every one of whose symbols appears A_{k+1}^n, so that they are on n cells of C_n. This carries out the reduction for $i=0$.

(*c*) Assume that the process goes through for any $M_{n'}$, $n'<n$, and also for all cells of more than $k-i=m$ dimensions of C_k'. I say that it goes through for all dimensionalities.

Consider an unreduced $E_m=A_0^{p_0}\cdots A_m^{p_m}$ of C_k', the p's being then all $<n-i=n-k+m$. To any $E_h=A_0^{p_0}\cdots A_m^{p_m}\ A_{m+1}^{p_{m+1}}\cdots A_h^{p_h}$ incident with E_m corresponds $E_{h-m+1}=A_{m+1}^{p_{m+1}}\cdots A_h^{p_h}$ which we sense so that if the first set of A's is, as we shall assume, an indicatrix of E_h, the last is one for E_{h-m-1}. The totality of these cells is a subcomplex of C_k which is an M_{n-m-1} homeomorphic to the boundary of a cell (No. 3). The incidence relations (boundary congruences) between corresponding cells are formally identical.

Let $q_0, q_1, \cdots, q_{n-m-1}$ be the set of integers in increasing order which together with $p_0, \cdots, p_m$ constitute the set $0, 1, \cdots, n$. The manifold M_{n-m-1} carries two defining complexes. The first C_{n-m-1} has the points A^{q_0} for vertices, the second C'_{n-m-1} the remaining points A^q. In fact C'_{n-m-1} is a regular subdivision of C_{n-m-1}, its vertices q_i being on i-cells of it. To show this it will suffice to examine the relation between those which correspond to q_0 and q_1. Let, for example, the sequence of the p's and q's in increasing order read $p_0, p_1, q_0, p_2, p_3, q_1, \cdots$, so that $p_0=0$, $p_1=1$, $q_0=2$, $\cdots, q_1=5, \cdots$. Consider now $E_2=A^0A^1A^3$ of C_n'. It is on a certain three-cell of C_n on which it is incident with exactly two cells $A^0A^1A_1^2A^3$ and

$A^0A^1A_2^2A^3$ of C_n'. Hence $E_{m+1}=A^0A^1A^3\cdots A^5A^{p_4}A^{p_5}\cdots A^{p_m}$ whose symbol is $(p_0,\cdots,p_3,q_1,p_4,\cdots,p_m)$ is incident with the two $(m+2)$-cells obtained by placing first A_1^2, then A_2^2 between A^1 and A^3. It follows that A^3 is on the one-cell $A_1^2A^3+A^3A_2^2$ of C_{n-m-1}. With the q's the statement is that A^{q_1} is on the one-cell $A_1^{q_0}A^{q_1}+A^{q_1}A_2^{p_3}$. The same reasoning applies to the other q vertices.

Observe that since the p's are all $< n-k+m$ the q's must include all integers from $n-k+m$ to m. Hence $q_{n-m-1}=n$, $q_{n-m-2}=n-1,\cdots,q_{n-k-1}=n-k+m$.

25. Let now $(p_0,\cdots,p_m;q_0',\cdots q_{k-m-1}')$ be the symbol for any k-cell of C_k' incident with E_m. The reduction being achieved by assumption for all cells of more than m dimensions, $(p_0,\cdots,p_m;q_i')$, the symbol of an $(m+1)$-cell, must possess an integer $\geqq n-k+m+1$ which can only be q_i'. Similarly $(p_0,\cdots,p_m;q_i',q_j')$ must include at least one integer $\geqq n-k+m+2$, and this can only be q_i' or q_j', etc. Finally, then, among the $k-m$ integers q_i' there must be one at least equal to each integer of the sequence $n-k+m+1, n-k+m+2,\cdots,n$. As they are all distinct and $\leqq n$ they constitute that very sequence and our cell has then the symbol $(p_0,\cdots,p_m;n-k+m+1,\cdots,n)$. The symbol of the corresponding cell of C'_{n-m-1} is $(n-k+m+1,\cdots,n)$.

26. Let $\bar{C}_k$ be the subcomplex of C_k' which is the sum of its cells incident with E_m. Since the latter is not on the boundary of C_k', on any E_{k-1} of $\bar{C}_k$, there are as many positively related incident k-cells as negatively related. Hence the complex of C'_{n-m-1} which corresponds to $\bar{C}_k$ is a cycle Γ_{k-m-1}. Since C'_{n-m-1} is homeomorphic to the boundary of a cell, Γ_{k-m-1} bounds on the complex, and in fact bounds a subcomplex C_{k-m} of C'_{n-m-1} (Coll. Lect., pp. 95, 118). Furthermore since the reduction to be proved applies by assumption to an M_{n-m-1}, C_{k-m} may be so reduced without changing its boundary that in the symbol of its s-cells not on the boundary Γ there appears $q_{n-k+s-1}$ or a higher q. But since the cells of Γ already satisfy this condition, it holds for all cells of C_{k-m} without exception. From this we conclude immediately that its $(k-m)$-cells have all the same symbol:

$$(q_{n-k-1},q_{n-k},\cdots,q_{n-m-1})\equiv(n-k+m,n-k+m+1,\cdots,n).$$

That it has this last simple form follows from the remark at the end of No. 24.

27. To C_{k-m} corresponds a subcomplex C_{k+1} of C_n' whose boundary is a Γ_k. The cells of this cycle incident with E_m constitute $\bar{C}_k$ so that E_m is not a cell of $C_k''=C_k'-\Gamma_k$. However, among the new cells of C_k'' are found those on k-cells of Γ_k not incident with E_m and we must examine these.

The symbol for any $(k+1)$-cell of C_{k+1} is $(p_0,\cdots,p_m;n-k+m,\cdots,n)$. The new k-cells introduced have then a symbol such as $(p_0,\cdots,p_{s-1},$

$p_{s+1}, \cdots, p_m; n-k+m, \cdots, n$). In the symbol of any cell of $m+j$ dimensions of its boundary will then appear $n-k+m+j$ or a higher integer. The new cells of m or more dimensions have then the desired behavior. Thus E_m and its incident cells have been replaced by a set of cells which fulfil all requirements. *The proof of our theorem is therefore complete.*

Remark. We have incidentally obtained the following interesting proposition: *Every cycle of a complex defining a manifold without boundary is homologous to a cycle on its dual.* More precisely *every* Γ_k *of* C_n *is homologous to a cycle whose k-cells have all the same symbol* $(n-k, n-k+1, \cdots, n)$.

28. We return to our approximation problem. The reduction of C_k has been obtained by adding the boundaries of $(k+1)$-cells incident with its k-cells and belonging to C_n. Let us replace C_n by a subdivision $\bar{C}_n$ with cells $<\epsilon$ and C_n' by a regular subdivision of $\bar{C}_n$. If we make the same reduction, we shall merely add to C_k the boundaries of $(k+1)$-cells within a distance of ϵ from the complex.

Applying this reduction to the approximating complex C_k' we find that we do not thereby disturb (21.1) or (21.2). The complex C_{k+1} is simply increased by cells as near as we please to C_k', hence to the approximated complex.

29. The essential property of C_k is that *the* S_{k-i} *of any* E_{k-i} *of the complex has the maximum degree of generality relatively to the space of the cell* C_n *that carries it.* We mean thereby that, by modifying the complex without changing its cellular structure, S_{k-i} may be brought into coincidence with an arbitrary neighboring S'_{k-i}. The weakest case is when E_{k-i} lies on an E_{n-i} of C_n with its vertices on $(n-k)$-cells of the boundary of E_{n-i}. Let A be a vertex of E_{k-i} on the cell E_{n-k}; S'_{k-i} will intersect the latter at a point B very near A. Impress upon A the rectilinear displacement AB, and similarly for the other $k-i$ vertices of E_{k-i}, leaving the remaining vertices of C_k' unchanged. There results an obvious deformation of C_k', into say C_k'', whereby S_{k-i} is brought into coincidence with S'_{k-i} thus proving our assertion.

The displacement of C_k' may be so carried out, and in a continuous way, that every point will describe a rectilinear path. Their locus is a $\bar{C}_{k+1} \equiv C_k'' - C_k'$. By adding this to (21.1) we see that C_k'' may take the place of C_k' with $C_{k+1}+\bar{C}_{k+1}$ in place of C_{k+1}. If S'_{k-i} is sufficiently near S_{k-i} conditions (a), (b), (c) of No. 21 may still be fulfilled. The reasoning in case E_{k-i} lies on a cell of more than $n-i$ dimensions is the same.

§5. Intersections of general complexes

30. To arrive at something significant, we must narrow down the problem once more. We replace then the second condition of No. 13 by the somewhat more sweeping

III. *Intersecting complexes must not meet the boundaries of one another.*

To complexes so restricted we shall ascribe definite cycles or Kronecker indices. Indices and cycles are fixed, in the sense that the former remain the same when we vary the approximating complexes or even the C_n by means of which we construct them, while the cycles remain homologous to themselves. We shall at first maintain C_n fixed and merely vary the polyhedral approximations, then examine the effect produced by a change of C_n.

31. Starting first with two complexes C_h, C_k always restricted as in No. 13, we approximate them as closely as we please by C_h', C_k' constructed as in No. 32 with the system of relations

$$(31.1) \qquad C_{h+1} \equiv C_h - C_h + C_h^0 ; \qquad C_h' \sim C_h + C_h^0 ;$$

$$(31.2) \qquad C_{k+1} \equiv C_k - C_k' + C_k^0 ; \qquad C_k' \sim C_k + C_k^0 ;$$

the various complexes have the same meaning as those of similar designation in §4. If the approximation is sufficiently fine, C_h' and C_k' will also fulfil the restrictive conditions I, III of Nos. 13 and 30. This we assume henceforth for all our approximating complexes.

It follows at once from No. 29 that we may so choose C_h' and C_k' that they satisfy the two conditions of No. 14 for a well defined intersection $C_h' \cdot C_k'$. Since the boundary of each complex does not meet the other, the intersection will be an l-cycle. It remains to be shown that this cycle is independent of the approximating complexes.

32. As a preliminary step let $\bar{C}_h'$ be another approximation whose intersection with C_k' is well defined. I say that

$$(32.1) \qquad C_h' \cdot C_k' \sim \bar{C}_h' \cdot C_k' .$$

We have now congruences such as (21.1)

$$(32.2) \qquad \bar{C}_{h+1} \equiv C_h + \bar{C}_h^0 - \bar{C}_h' ,$$

$$(32.3) \qquad C_{h+1} \equiv C_h + C_h^0 - C_h'$$

with C_h^0, $\bar{C}_h^0$ very near the boundary of C_h and C_{h+1}, $\bar{C}_{h+1}$ very near the complex itself. Their approximation is in fact assumed such throughout that none of these complexes meets the boundary of C_k'. We have then from (32.2) and (32.3)

$$\bar{C}_{h+1} - C_{h+1} \equiv C_h' - \bar{C}_h' - (C_h^0 - \bar{C}_h^0) .$$

To the complex at the left we may apply everything said previously for C_k with the following result. There exists a polyhedral complex C_{h+1}' very

near it, having a well defined intersection with C_k' and whose boundary is $C_h'-\bar{C}_h'$ plus a certain h-complex very near $C_h^0-\bar{C}_h^0$ and therefore not meeting C_k'. Hence by (16.1)

$$C_{h+1}' \cdot C_k' \equiv (C_h'-\bar{C}_h') \cdot \bar{C}_k' \sim 0 \qquad (\text{mod } M_n) ,$$

from which (32.1) follows.

33. Let now $\bar{C}_h'$, $\bar{C}_k'$ be two approximations with a well defined intersection. In order to show that the intersections are independent of the particular polyhedral approximations provided they have a well defined intersection, we must prove that

$$\bar{C}_h' \cdot \bar{C}_k' \sim C_h' \cdot C_k' . \tag{33.1}$$

By a slight displacement such as is used in No. 29, we may replace C_h' by a complex C_h'' with a well defined intersection with both C_k' and $\bar{C}_k'$. All that is necessary is to replace throughout C_k' by $C_k'-\bar{C}_k'$. We have then according to the preceding number

$$C_h' \cdot C_k' \sim C_h'' \cdot C_k' ; \quad \bar{C}_h' \cdot \bar{C}_k' \sim C_h'' \cdot \bar{C}_k' ;$$

$$C_h'' \cdot C_k' \sim C_h'' \cdot \bar{C}_k' ,$$

whence (33.1) follows.

Regarding the Kronecker indices, Corollary IV at the end of §3 yields at once for $h+k=n$

$$((C_h'-\bar{C}_h'-(C_h^0-\bar{C}_h^0)) \cdot C_k') = 0 ,$$

and as C_k' does not intersect $C_h^0-\bar{C}_h^0$,

$$(C_h' \cdot C_k') = (\bar{C}_h' \cdot C_k')$$

and the rest is as before.

34. The extension to more than two intersecting complexes is easy. With obvious notations we must show that

$$C_h' \cdot C_k' \cdot \cdots \cdot C_l' \sim \bar{C}_h' \cdot \bar{C}_k' \cdot \cdots \cdot \bar{C}_l' .$$

Introduce C_h'' in general position as to C_k', $\bar{C}_k'$, $\cdots$, C_l', $\bar{C}_l'$. As above it may be shown that in the homology C_h' and $\bar{C}_h'$ may both be replaced by C_h'', the process continuing in an obvious way. The treatment for indices is the same.

35. As is natural we denote the cycles and indices defined by means of our approximations as $C_h \cdot C_k \cdot \cdots \cdot C_l$, or $(C_h \cdot C_k \cdot \cdots \cdot C_l)$. These symbols have the same properties as those for polyhedral complexes them-

selves. Those pertaining to permutation of complexes hold obviously, others less so. We shall examine them in turn, with particular reference to our future needs.

I. *Associative law.* The scheme of the proof is sufficiently illustrated with three complexes and if we show that

$$C_h \cdot (C_k \cdot C_l) \sim C_h \cdot C_k \cdot C_l .$$

By definition $C_k \cdot C_l$ and $C_h \cdot C_k \cdot C_l$ are the polyhedral cycles $C_k' \cdot C_l'$ and $C_h' \cdot C_k' \cdot C_l'$. In the approximation that leads to the determination of $C_h \cdot (C_k' \cdot C_l')$, the cycle in parentheses can be taken as its own approximation. Hence the cycle at the left in the homology is by definition $C_h' \cdot (C_k' \cdot C_l')$ and we are back to the case of polyhedral complexes in general position, for which the law holds.

II. *Distributive law.* We wish to show that, say,

$$C_h \cdot (C_k + \bar{C}_k) \sim C_h \cdot C_k + C_h \cdot \bar{C}_k .$$

On examining the two successive approximations of §4 it will be seen that $C_k' + \bar{C}_k'$ is an approximation for $C_k + \bar{C}_k$. If each of the two primed complexes has a well defined intersection with C_h' so has their sum. Hence the left side is by definition $C_h' \cdot (C_k' + \bar{C}_k')$. As the terms at the right are also defined by means of the primed symbols, we are again back to the case of polyhedral complexes where the law holds.

The two preceding proofs hold without modification for the Kronecker index.

III. *If $C_h, C_k, \cdots, C_l$ do not actually have a common point, then*

$$C_h \cdot C_k \cdots C_l \sim 0 , \text{ or } (C_h \cdot C_k \cdots C_l) = 0 .$$

For then the primed complexes may be taken without any common point and everything is once more reduced to the known case of polyhedral complexes.

IV. *Let C_h bound $\bar{C}_{h+1}$ such that it is the only one of the set $\bar{C}_{h+1}, C_k, \cdots, C_l$ whose boundary may have points in common with the other complexes. Then*

$$C_h \cdot C_k \cdots C_l \sim 0 \text{ or } (C_h \cdot C_k \cdots C_l) = 0 .$$

From (31.1) follows $\bar{C}_{h+1} - C_{h+1} \equiv C_h'$, for C_h^0 is now absent since C_h is a cycle. Moreover C_h being a subcomplex of $\bar{C}_{h+1}$ the sequence $C_h, C_k, \cdots, C_l$ behaves like that of the statement. Let us approximate $\bar{C}_{h+1} - C_{h+1}$ by, say, C_{h+1}', in our usual manner, which may be done without changing C_h since it is on the boundary (property (*c*), No. 21). Since C_{h+1} is very near C_h'

it will be seen that the primed sequences corresponding to the two considered above behave as they do. The generalization of (16.1) gives here

$$C'_{h+1} \cdot C_h' \cdot \cdots C_l' \equiv C_h' \cdot C_k' \cdot \cdots \cdot C_l'$$

from which IV follows.

V. *Let* $\Gamma_h \approx 0$ (*i. e., some multiple of* $\Gamma_h \sim 0$). *Then*

$$\Gamma_h \cdot \Gamma_k \cdot \cdots \Gamma_l \approx 0 \text{ or } (\Gamma_h \cdot \Gamma_k \cdot \cdots \Gamma_l) = 0 .$$

This is an immediate corollary of IV. In both IV and V it is of course not at all necessary that the complex or cycle singled out be the first.

VI. *In* C_h *we may suppress any subcomplex not intersecting* $C_k, \cdots, C_l$, *without affecting intersection cycle or index.*

This is an immediate but important corollary of II and III.

36. Before we proceed with a thorough examination of the effect of passing from C_n to a new defining $\bar{C}_n$, let us observe that *instead of approximating to* C_h *by means of* $\bar{C}_n$ *it is sufficient to do this for* C_h'. Indeed, let $\bar{C}_h'$ be the approximation to C_h' by means of $\bar{C}_n$. We shall have a congruence such as (21.1):

$$\bar{C}_{h+1} \equiv C_h' - \bar{C}_h' + \bar{C}_h^0 ,$$

with $\bar{C}_{h+1}$ very near C_h' and $\bar{C}_h^0$ very near its boundary. Add this congruence to the first of (31.1) (which is the same as (21.1) with k replaced by h):

$$(C_{h+1} + \bar{C}_{h+1}) = C_h - \bar{C}_h' + (C_h^0 + \bar{C}_h^0) .$$

This is analogous to (21.1) with $\bar{C}_h'$ as the approximation to C_h. As the congruences such as (21.1) plus the structure of the approximating complexes themselves were alone used in defining the intersection cycles and indices and deriving their properties, our assertion is proved.

§6. Proof that intersection cycles and indices are invariant when the defining complex is changed*

37. For indices the invariance is conditioned upon a certain simple sense convention. Let C_n, $\bar{C}_n$ be any two defining complexes. By practically the same reasoning as Veblen's in *Colloquium Lectures*, pp. 101, 102, we may show that one of the two complexes $C_n \pm \bar{C}_n$ is a bounding cycle, but not both (loc. cit., p. 120). We shall assume in the future that *the complexes are so oriented that* $C_n - \bar{C}_n$ *is the bounding cycle.* Once any particular complex has been assigned an orientation, a definite one follows for the rest.

* A first type of proof is outlined in my second Proceedings note. Just as it appeared in print I discovered the much simpler treatment embodied in this section.

We must examine if the present convention agrees with our previous mode of sensing a subdivision C_n' of C_n, where it will be remembered a cell and its subdivisions were always so sensed as to have a common indicatrix. It is clearly sufficient to consider the case where C_n is a simplex with a simplicial subdivision:

$$C_n' = \Sigma_n^1 + \cdots + \Sigma_n^p .$$

Let the first q simplexes, but no others, have an $(n-1)$-simplex on a given simplex Σ_{n-1} of Σ_n, namely Σ_{n-1}^i for Σ_n^i, with A and A^i as the vertices of Σ_n and Σ_n^i not on Σ_{n-1} or Σ_{n-1}^i. We take the orientations such that A followed by the vertices of Σ_{n-1} corresponds to the same as A_i followed by those of Σ_{n-1}^i.

In order that $\Sigma_n - C_n'$ be a cycle, it is necessary and sufficient that the boundaries on Σ_{n-1} cancel, or that

$$\Sigma_{n-1} - (\Sigma_{n-1}^1 + \cdots + \Sigma_{n-1}^q)$$

be a cycle. This reduces the verification from n to $n-1$, hence ultimately to $n=1$ for which it is immediate.

38. Let us return to two arbitrary defining complexes C_n, $\bar{C}_n$. Our customary approximations, when applied to $\bar{C}_n$ by means of C_n, cannot go farther than what is yielded by the Alexander-Veblen process. In this case it comes down to this: We subdivide C_n and $\bar{C}_n$ into C_n' and $\bar{C}_n'$, then establish a correspondence T whereby to each cell of C_n' is assigned a unique one of C_n'. The n-cells of C_n' are each covered positively by k_2 cells corresponding to positive cells of $\bar{C}_n'$ and negatively by k_1 such cells. Furthermore it is a property of the correspondence that $C_n' - T(\bar{C}_n')$ bounds, hence $k_2 - k_1$ is fixed for all n-cells of C_n, else this complex would have boundary cells exterior to the boundary of M_n and would not be a cycle. Moreover

$$T(\bar{C}_n') \sim \bar{C}_n' \sim \bar{C}_n \sim C_n \sim C_n' ;$$

therefore

$$C_n' - T(\bar{C}_n') = (1 + k_1 - k_2) C_n' \sim 0 .$$

When a subcomplex of a C_n bounds it bounds also such a subcomplex (Coll. Lect. p. 118). But C_n' has no $(n+1)$-cells, hence this homology can only be true if $1 + k_1 - k_2 = 0$, $k_2 - k_1 = 1$. *Thus $T(\bar{C}_n')$ covers C_n' exactly once.* The importance of this will appear later.

39. **Invariance of the index.** It will be established by means of (18.1) that

$$(C_h \cdot \Gamma_{n-h}) = (-1)^h \cdot (\Gamma_{h-1} \cdot C_{n-h+1}), \tag{39.1}$$

where the conditions of No. 15 must be satisfied. Here the intersecting complexes are the two extremes, the cycles being their boundaries. We have, then, given C_h, C_{n-h}, their approximations C_h', C'_{n-h} by means of a first defining complex C_n, and $\bar{C}_h'$, $\bar{C}'_{n-h}$ by a second $\bar{C}_n$, the latter being, if we wish, approximations to C_h' and C'_{n-h} rather than to the given complexes (No. 37). We denote the indices as to C_n by the usual round parentheses, those as to $\bar{C}_n$ by square parentheses, and our object is to show that

$$(C_h' \cdot C'_{n-h}) = [C_h' \cdot C'_{n-h}] . \tag{39.2}$$

Assuming first that neither h nor $n-h$ are zero, we shall replace (39.2) by a similar formula with $h-1$ in place of h. This will allow us to reduce everything to the case $h=0$ for which the proof is simple.

40. Suppose that there exists a polyhedral C'_{n-h+1} (until further notice straightness and the like are defined by reference to C_n) whose boundary

$$\Gamma'_{n-h} = C'_{n-h} + C''_{n-h}$$

where C'_h and C''_{n-h} do not meet. Then first

$$(C'_h \cdot C'_{n-h}) = (C'_h \cdot \Gamma'_{n-h}) .$$

Also assuming all due conditions satisfied, and denoting by Γ'_{h-1} the boundary of C'_h, we find from (39.1)

$$(C'_h \cdot C'_{n-h}) = (-1)^h \cdot (\Gamma_{h-1} \cdot C'_{n-h+1}) . \tag{40.1}$$

The necessary conditions are fulfilled with ease. According to our very construction of approximating complexes, C'_h and C'_{n-h} intersect in a finite number of points, any one of which, say A_i, is on cells of maximum dimensionality of the two complexes and of C_n. Let in particular E^i_{n-h} be that of C'_{n-h}. We can construct a simplicial E^i_{n-h+1} with E^i_{n-h} on its boundary and positively related to it. Let this last cell count m_i times for C'_{n-h}. We choose

$$C'_{n-h+1} = \sum m_i E^i_{n-h+1} ;$$

C''_{n-h} is now what is left of the boundaries of the cells at the right when the cells $m_i E^i_{n-h}$ are removed from them. We may therefore manifestly so choose the cells E^i_{n-h+1} that C_{n-h} will contain no A point.

Without prejudice to what precedes, the cells of C''_{n-h} may be brought as near as we please to C'_{n-h}, hence we may so construct C'_{n-h+1} that C''_{n-h} intersects C_h in a finite number of points (none an A), each on h-, $(n-h+1)$- and n-cells of C'_h, C'_{n-h+1} and C_n. Let B be such a point. Remove from C_h'

a simplicial cell E_h containing B. If it is sufficiently small, it will contain no A and its bounding $(h-1)$-cells will intersect C'_{n-h+1} each at a single point of the cell that carries B. Operate similarly for all B's and let the new complex still be denoted by C'_{h-1} and its boundary by Γ'_{h-1}.

If the cells of C'_{n-h+1} and those removed from the initial C_h' are adequately small, the intersection of the new C'_h with C'_{n-h+1} will be a sum of isolated rectilinear segments each on an n-cell of C_n, and as between the two complexes the various restrictions of §3 are verified. Since the new C_h' does not meet C''_{n-h}, it gives rise to the same index $(C'_h \cdot C'_{n-h})$ as the initial one, so that we are assured that (40.1) holds with C_h' as the new approximation to C_h and Γ'_{h-1} as its boundary.

41. At this stage we introduce the second defining complex $\bar{C}_n$. All approximations by means of it are to be denoted by the same symbols barred.

We first construct $\bar{\Gamma}'_{h-1}$, $\bar{C}'_{n-h}$, $\bar{C}''_{n-h}$. On examining our approximating processes, it is seen that the sum of the last two complexes is a suitable $\bar{\Gamma}'_{n-h}$. The cycle $\bar{\Gamma}'_{h-1}$ bounds $C'_h+C^0_h$, where the second complex, introduced by the approximation, is as near as we please to Γ'_{h-1}. Since C'_h and C'_{n-h} were constructed in general relative position, the second does not meet the boundary Γ'_{h-1} of the first. With approximations sufficiently fine, it will not meet C^0_h either. Hence (No. 35, VI)

$$[(C'_h+C^0_h) \cdot C'_{n-h}] = [C'_h \cdot C'_{n-h}] .$$

Therefore in place of approximating C_h' we may approximate $C_h'+C^0_h$. To avoid more notations than necessary, we shall denote that approximation by $\bar{C}_h'$.

Similarly $\bar{\Gamma}'_{n-h}$ bounds a complex $C'_{n-h+1}+C^0_{n-h+1}$, the latter very near the cycle and therefore not intersecting Γ'_{h-1}, hence, as above, its approximation may take the place of that of C'_{n-h+1} and will then be denoted by $\bar{C}'_{n-h+1}$.

As previously, we may choose the approximations $\bar{C}'_h$ and $\bar{C}'_{n-h+1}$ such as to fulfil the conditions of §3 (Nos. 13, 15), for a well defined intersection. Even for $h=1$ is this the case. $\bar{C}_1'$ may then be so chosen that the points of Γ_0' are approximated by any in their vicinity. We have therefore in all cases where $h>0$

$$[\bar{C}'_h \cdot \bar{C}'_{n-h}] = (-1)^h [\bar{\Gamma}'_{h-1} \cdot \bar{C}'_{n-h+1}] .$$

By the definition of the index, each expression defines that of the approximated complexes, which are here the primed complexes. Hence

$$[C'_h \cdot C'_{n-h}] = (-1)^h \cdot [\Gamma'_{h-1} \cdot C'_{n-h+1}] .$$

42. From (40.1) and (41.1) follows that in place of (39.2) we need only prove

$$(\Gamma'_{h-1} \cdot C_{n-h+1}) = [\Gamma'_{h-1} \cdot C_{n-h+1}] ,$$

of the same type but with h replaced by $h-1$. Proceeding thus if necessary we shall reduce h to zero. The last step will consist in replacing C_1' by its set of terminal points Γ_0', the signs affixed to them being $+$ or $-$ according as they are positively or negatively related to the complex. $\overline{\Gamma}_0'$ will then consist of points in the same number as those of Γ_0', very near to them and with same signs attached.

For the other complex, we now have a C_n' and by subdividing C_n if necessary, we may assume that C_n' is a subcomplex of it. Furthermore, (No. 41), the points of Γ_0' may all be chosen on n-cells of C_n'.

Owing to the distributive law, we may assume in the last analysis that we have a unique point A with a definite sign affixed, and a unique cell E_n of C_n carrying the point A. We have seen that the sign of the point is to be independent of the approximation; let us assume that it is $+$. In the other possible case, the reasoning would be the same with perhaps some signs changed. We must then show that

$$(A \cdot E_n) = +1 = [A \cdot E_n] .$$

Since A is not on $C_n - E_n$, we may add the last complex to E_n without affecting the indices. We have to prove then, that

$$[A \cdot C_n] = +1 .$$

We may choose A on E_n not a cell of less than n dimensions of $\overline{C}_n$ without affecting our indices. Let A be its own approximation by means of $\overline{C}_n$. The approximation of C_n by means of $\overline{C}_n$ will be the sum of the cells of a certain subdivision of $\overline{C}_n$ (No. 40), which is the same as $\overline{C}_n$ itself. Finally then

$$[A \cdot C_n] = [A \cdot \bar{C}_n] = +1 ,$$

which completes the proof of the invariance of the index of two complexes.

43. The extension to the index of several complexes can be made along the same lines. The essential thing is to obtain a relation analogous to (40.1). It will be sufficient to derive it for three complexes C_h, C_k, C_l, $h+k+l=2n$, with the usual conditions as to their boundaries. Their approximations will then satisfy conditions analogous to those of No. 15. With obvious notations, let C'_{h+1} have for boundary $\Gamma'_h = C'_h + C''_h$ with C''_h not intersecting C_k' nor C_l'. As a matter of fact, it would be sufficient to have it not intersecting $C_k' \cdot C_l'$. Then

$$(C_h' \cdot C_k' \cdot C_l') = (\Gamma_h' \cdot C_k' \cdot C_l') . \tag{43.1}$$

Also by (16.1) and with Γ'_{k-1} the boundary of C_k',

$$C'_{h+1} \cdot C'_k \equiv (-1)^{n-k} \cdot \Gamma'_h \cdot C_k + C'_{h+1} \cdot \Gamma'_{k-1} ,$$

it being granted that conditions of No. 15 are duly fulfilled. If we now further assume that the boundary of C_l' does not meet the complex at the left, then (No. 19)

$$(-1)^{n-k}(\Gamma'_h \cdot C'_k \cdot C'_l) + (C'_{h+1} \cdot \Gamma'_{k-1} \cdot C'_l) = 0 .$$

Hence by (43.1)

$$(C'_h \cdot C'_k \cdot C'_l) = (-1)^{n-k-1}(C'_{h+1} \cdot \Gamma'_{k-1} \cdot C'_l) .$$

Similarly, with notations whose meaning is transparent,

$$C'_h \cdot C'_k \cdot C'_l = (-1)^{n-l-1} \cdot (C'_h \cdot C'_{k+1} \cdot \Gamma'_{l-1}) .$$

Thus we can raise both h and k at the expense of l, until $h=k=n$, $l=0$. From this point on the details of the discussion differ in no material way from the case of two complexes and need not be given here.

We have then proved that *the index of any number of complexes, when any exists, is independent of the defining C_n of M_n and the related straightness.* It has of course the various properties given in No. 36.

44. **Invariance of the intersection cycle.** The reasoning is exactly the same whatever the number of complexes, so we need only take two, C_h, C_k. This time, with our previous notations, we have to show that

$$C'_h \cdot C'_k \sim \bar{C}'_h \cdot \bar{C}'_k .$$

The cycle $C_h' \cdot C_k'$ is carried by a C_l, of the simplest type of No. 2, sum of the distinct simplicial l-cells of a decomposition of the cycle into such cells. Let these cells be $E_l^1, \cdots, E_l^p$, oriented each in a definite way. We shall have

$$C_l = \sum E_l^i ; \quad C'_h \cdot C'_k = \sum t_i E_l^i ,$$

where the t's are non-zero integers.

Since the barred complexes are as near as we please to the non-barred, $\bar{C}_h' \cdot \bar{C}_k'$ is as near as we please to C_l. Hence (No. 22) the latter carries a Γ_l, subcomplex of a sufficiently fine subdivision, such that $\bar{C}_h' \cdot \bar{C}_k' - \Gamma_l$ bounds a C_{l+1} whose points are as near as we please to C_l. But Γ_l is homologous, mod C_l, to a subcomplex of C_l itself (Coll. Lect., p. 120). Hence we

may assume that Γ_l itself is such a subcomplex without weakening the assertion as to C_{l+1}. We shall have

$$\bar{C}_k' \cdot \bar{C}_k' \sim \sum s_i E_l^i = \Gamma_l .$$

Owing to our usual approximating procedure, each E_l^i is on an n-cell of C_n. Hence, just as if C_n were an S_n, we can construct a simplicial E_{n-l} that intersects C_l at a unique point on E_l^i in such manner that

$$(E_l^i \cdot E_{n-l}) = +1 ; \qquad (E_l^j \cdot E_{n-l}) = 0 , \qquad j \neq i .$$

By taking E_{n-l} sufficiently small, we may dispose of it so that its boundary does not meet C_l, while the cell itself does not meet the boundaries of C_k' or C_k'. This is due to the fact that the latter do not intersect C_l while E_{n-l} is very near to this complex.

Let us now approximate by means of $\bar{C}_n$. We have two congruences such as in No. 31:

$$\bar{C}_{k+1} \equiv C_k' - \bar{C}_k' + \bar{C}_k^0 ,$$

$$\bar{C}_{k+1} \equiv C_k' - \bar{C}_k' + C_k^0 .$$

When the approximation is carried sufficiently far,

(a) the boundary of E_{n-l} will not meet C_{l+1};

(b) it will not go through any intersection of $\bar{C}_{k+1}$ with C_k' or $\bar{C}_k'$, nor of $\bar{C}_{k+1}$ with C_k' or $\bar{C}_k'$;

(c) E_{n-l} will not meet $\bar{C}_k^0$ nor $\bar{C}_k^0$.

Since C_{l+1} is as near as we please to C_l, (a) follows from the fact that the boundary does not intersect C_l. Now were the first of (b) untrue, since $\bar{C}_{k+1}$ is as near as we please to C_k, it would only mean that the boundary of E_{n-l} goes through a point of $C_k' \cdot C_k'$, and hence meets C_l which carries this cycle, and this is not the case. Similarly for the rest of (b). As to (c), it follows at once from the fact that the cell does not meet the boundaries of C_k' and C_k' to which $\bar{C}_k^0$ and $\bar{C}_k^0$ are very near.

Owing to (a) and to IV and I of No. 36, we have

$$((\Gamma_l - \bar{C}_k' \cdot \bar{C}_k') \cdot E_{n-l}) = 0 ;$$

therefore

$$\sum s_j (E_l^j \cdot E_{n-l}) = s_i = (\bar{C}_k' \cdot \bar{C}_k' \cdot E_{n-l}) .$$

Also

$$(C_k' \cdot C_k' \cdot E_{n-l}) = \sum t_j (E_l^i \cdot E_{n-l}) = t_i .$$

Again by IV and the first of (b),

$$((C_k' - \bar{C}_k' + \bar{C}_k^0) \cdot C_k' \cdot E_{n-l}) = 0 .$$

From (c) we have

$$(\bar{C}_h^0 \cdot C_k' \cdot E_{n-l}) = 0 .$$

Therefore

$$(C_h' \cdot C_k' \cdot E_{n-l}) = (\bar{C}_h' \cdot C_k' \cdot E_{n-l}) .$$

Similarly from the last of (b) and (c),

$$(\bar{C}_h' \cdot (C_k' - \bar{C}_k' + \bar{C}_k^0) \cdot E_{n-l}) = 0 .$$

Therefore

$$(\bar{C}_h' \cdot C_k' \cdot E_{n-l}) = (\bar{C}_h' \cdot \bar{C}_k' \cdot E_{n-l}) .$$

By comparing we have at once

$$t_i = (C_h' \cdot C_k' \cdot E_{n-l}) = (\bar{C}_h' \cdot \bar{C}_k' \cdot E_{n-l}) = s_i .$$

From this follows finally

$$C_h' \cdot C_k' \sim \bar{C}_h' \cdot \bar{C}_k' ,$$

and our invariance proof is complete.

45. Intersection cycles or Kronecker indices are then the same whatever the defining complex of M_n from which they are derived, provided as regards indices that various complexes have their orientations suitably related. Cycles and indices have all the properties established in No. 35 and the effect of a permutation of complexes in the symbols is the same as for cells.

§7. Fundamental sets on an orientable M_n without boundary

46. Let $\Gamma_h^i (i = 1, 2, \cdots, p_h)$, Γ_{n-h}^j $(i = 1, 2, \cdots, p_{n-h})$ be two fundamental sets (No. 7) for the cycles of the same dimensions, and consider the matrix

$$(46.1) \qquad \| (\Gamma_h^i \cdot \Gamma_{n-h}^j) \| .$$

Any other fundamental set, say for the dimensionality h, can be derived from a given one by applying to its cycles a transformation of determinant unity. This is at once derivable from the result established by Veblen, loc. cit., p. 117. By such a transformation we mean that every cycle of the new set will be homologous to a sum of cycles of the old, the matrix of the coefficients of the homologies being ± 1.

According to the distributive law for indices, if we change fundamental sets the above matrix is merely multiplied to the right or to the left by a matrix of determinant ± 1. *Therefore the invariant factors of* (46.1) *are independent of the two fundamental sets*, and in fact *they are all equal to unity*,

an important result just proved by Veblen,* which we shall also derive later in a new way (No. 65). It follows from this and from Poincaré's theorem on the equality of the numbers R_h, R_{n-h}, that we may select our sets so that

(*a*) for $j > R_h$, Γ_h^j and Γ_{n-h}^j are zero-divisors;

(*b*) if $h \neq n/2$ or if $h = n/2$ and $n/2$ is even,

$$(\Gamma_h^i \cdot \Gamma_{n-h}^i) = \left\{ \begin{matrix} +1 & \text{if } h \leqq n/2 \\ (-1)^{(n+1)h} & \text{if } h > n/2 \end{matrix} \right\} i \leqq R_h ,$$

all other indices being zero;

(*c*) if $h = n/2$ and $n/2$ is odd, when (46.1) is an alternate matrix,

$$(\Gamma_{n/2}^{2i-1} \cdot \Gamma_{n/2}^{2i}) = -(\Gamma_{n/2}^{2i} \cdot \Gamma_{n/2}^{2i-1}) = +1 , \; i \leqq \tfrac{1}{2} R_{n/2}$$

with all other indices again zero. In this case, of course, *since the matrix is alternate and of rank $R_{n/2}$ this last integer is necessarily even*, a generalization of the well known result for two-dimensional manifolds.

The possibility of choosing the sets as above is based upon well known theorems on the reduction of matrices with integer terms.†

Fundamental sets of the type just described will be called *canonical*.

PART II. TRANSFORMATION OF MANIFOLDS

§1. PRODUCT COMPLEXES‡

47. Let E_p, E_q be two cells, A a point of E_p or of its boundary, B a similar point for E_q. Consider the set of couples A, B which by definition vary continuously if either A or B so varies. I say that the set is an E_n, $n = p+q$, plus its boundary. This follows at once from the fact that if $x_1, x_2, \cdots, x_n$ are cartesian coördinates for an S_n, E_p, E_q with their boundaries, then the sets in question are respectively homeomorphic to the following three sets:

$$0 \leqq x_i \leqq 1, \quad i \leqq p; \quad x_{p+j} = 0 ;$$

$$x_i = 0, \; i \leqq p; \quad 0 \leqq x_{p+j} \leqq 1 ;$$

$$0 \leqq x_i \leqq 1, \quad i = 1, 2, \cdots, n .$$

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* These Transactions, vol. 25 (1923), p. 540. See in the same connection my Monograph already quoted, p. 13. In two recent notes of the Proceedings of the National Academy of Sciences, vol. 10 (1924), pp. 99–103, J. W. Alexander generalizing this notion has been led to a new set of topological invariants.

† See the expository paper by Veblen and Franklin in the Annals of Mathematics, ser. 2, vol. 23.

‡ The term and corresponding notation are due to E. Steinitz, Sitzungsberichte der Berliner mathematischen Gesellschaft, vol 7 (1908). See also in this connection H. Tietze, Abhandlungen des mathematischen Seminars zu Hamburg, vol. 2 (1923), p. 37, H. Künneth, Mathematische Annalen, vol. 90 (1923), p. 65, and my paper in these Transactions, vol. 22 (1921), p. 362.

48. Let Π_p, Π_q be polyhedra in spaces S_r, S_t, by means of which straightness and distances are defined for E_p and E_q and let $x_1, \cdots, x_r$ and $y_1, \cdots, y_t$ be cartesian coördinates for S_r and S_t. We may refer an S_{r+t} to the set of cartesian coördinates x_i, y_j. Then if a point (x) describes Π_p and a point (y) describes Π_q the corresponding point (x, y) of S_{r+t} describes a Π_n homeomorphic to E_n plus its boundary. The geometry of this is very elementary and we leave it to the reader. This Π_n shall be chosen as basis for straightness and distances on E_n and its boundary.

The *point* A, B of E_n or its boundary shall be designated henceforth by $A\times B$. Let $AA_1\cdots A_p$, $BB_1\cdots B_q$ be indicatrices of E_p and E_q. We shall agree to sense E_n by the indicatrix

$$A\times B\ A_1\times B\cdots A_p\times B\ A\times B_1\cdots A\times B_q\ ,$$

and the cell so sensed shall be denoted by $E_p\times E_q$ and called *product* of the two cells, its *factors*.* The notation $A\times B$ merely corresponds to $p=q=0$. At once, we have

$$E_p\times E_q=-(-E_p)\times E_q=-E_p\times(-E_q)=(-1)^{pq}E_q\times E_p\ .$$

Let a^i_{p-1}, b^j_{q-1} be the boundary cells of E_p and E_q, so sensed that

$$E_p\equiv\sum a^i_{p-1}\ ,\qquad E_q\equiv\sum b^j_{q-1}\ .$$

Then

$$E_p\times E_q\equiv\sum\epsilon_i a^i_{p-1}\times E_q+\sum\eta_j E_p\times b^j_{q-1}\ , \tag{48.2}$$

where ϵ_i and η_j are ±1. To determine their actual value assume that $BB_1\cdots B_{q-1}$ is on b^j_{q-1}. Then $(-1)^q\cdot BB_1\cdots B_{q-1}$ is an indicatrix of b^j_{q-1} as boundary cell of E_q. That of $E_p\times b^j_{q-1}$ is

$$(-1)^q A\times B\ A_1\times B\cdots A_p\times B\ A\times B_1\cdots A\times B_{q-1}\ .$$

Comparing with the indicatrix of $E_p\times E_q$ we find $\eta_j=(-1)^q$. If we interchange E_p and E_q we shall find by applying this very relation $(-1)^q\ \epsilon_i=(-1)^q$, hence $\epsilon_i=1$, so that finally (48.2) becomes

$$E_p\times E_q\equiv\sum a^i_{p-1}\times E_q+(-1)^p\sum E_p\times b^j_{q-1}\ , \tag{48.3}$$

which describes the boundary of the product.

49. Let now C_p, C_q be any complexes, a^i_p a generic p-cell of the first, b^j_q one of the second. We define their product by the relation

$$C_p\times C_q=\sum a^i_p\times b^j_q\ .$$

* Concerning this symbolic product see the footnote to No. 7.

By applying (48.3) to each term at the right we find

$$C_p \times C_q \equiv \sum a_{p-1}^i \times b_q^j + (-1)^p \sum a_p^i \times b_{q-1}^j \tag{49.1}$$

where only the boundary $(p-1)$- and $(q-1)$-cells appear at the right with the very orientation that they possess in the boundary congruences of the complexes. Hence if C_{p-1} and C_{q-1} are these boundaries,

$$C_p \times C_q \equiv C_{p-1} \times C_q + (-1)^p C_p \times C_{q-1} . \tag{49.2}$$

From (49.2) we have the following:

I. *The product is orientable* (in the sense that a manifold is) *if and only if each factor is.*

II. *The product of two manifolds is a manifold.*

III. *The product of two complexes is without boundary if, and only if, the complexes themselves have none.* Observe that all this can be extended to singular complexes, hence

IV. *The product of two cycles of the factors is a cycle of the product.*

From polyhedra of the factors we derive as in No. 48 one for the product, hence corresponding definitions of straightness and distances.

50. Every Γ_k of $C_p \times C_q$ is homologous to a polyhedral cycle Γ_k' (Coll. Lect., p. 120). Let α be a point of the cell $a_p^i \times b_q^j$ not on Γ_k'. Draw a rectilinear segment from α to every point β of Γ_k' on the cell, to its intersection at γ with the boundary. The set of all segments $\beta\gamma$ constitutes a polyhedral C_{k+1}. The effect of subtracting its boundary from Γ_k' is to reduce the latter to a similar cycle without any points on $a_p^i \times b_q^j$. On proceeding thus with all such cells, then with the C_{n-1} made up with the cells of less than n dimensions of $C_p \times C_q$, and so on, we shall reduce Γ_k' to a homologous polyhedral cycle whose points are all on cells of at most k dimensions of the complex. From the theorem of the Colloquium Lectures just recalled, as applied to the C_k made up of the cells of at most k dimensions of our complex, follows that the reduced cycle, which we still call Γ_k', is a sum of k-cells of $C_p \times C_q$. This may also be shown by remarking that if it has points on an E_k the whole cell must belong to it, else it would have a boundary on it.

We conclude then that *every cycle of a product is homologous to a sum of products of cells of its two factors.*

Suppose that such a Γ_k bounds on $C_p \times C_q$. We may apply to the complex that it bounds the identical reasoning, the operations made not affecting the boundary, and we conclude that *if a cycle, sum of products of cells of the factors, bounds on the product, it bounds a complex which is also such a sum.*

Henceforth we shall use the notation γ_k, δ_k for the k-cycles of C_p and C_q, keeping Γ_k itself for those of the product $C_p \times C_q$.

51. As shown by Veblen* every E_k of C_p is expressible as a sum of multiples of certain cycles γ_p^i and certain cells a_k^j. Of course no sum of these particular a's is a γ, hence every γ_k of C_p is a sum of multiples of the cycles γ_k^j alone. It is not ruled out that some of the latter bound, but their set may be so chosen as to include a *fundamental set* for the k-cycles. There exists of course a similar set δ_k^i, b_k^j for C_q.

52. THEOREM. *The k-cycles which are products of cycles taken from fundamental sets of the factors constitute a fundamental set for the product.*

According to No. 50, for any Γ_k we have

$$\Gamma_k \sim \sum \gamma_\mu^i \times \delta_{k-\mu}^i + \sum \gamma_\mu^i \times b_{k-\mu}^j + \sum a_\mu^i \times \delta_{k-\mu}^j + \sum a_\mu^i \times b_{k-\mu}^j .$$

We must express the fact that the right side has no boundary. The boundary of $a_\mu^i \times b_{k-\mu}^j$ is a sum of terms $\gamma_{\mu-1} \times b_{k-\mu}^j$, $a_\mu \times \delta_{k-\mu-1}$ where $\gamma_{\mu-1}$, $\delta_{k-\mu-1}$ are sums of cycles described in the preceding number. The boundaries of an $a \times \delta$ or of a $\gamma \times b$ are both of type $\gamma \times \delta$. Hence the boundaries of the terms in the fourth sum can only cancel each other, which compels the sum to be similar to the third. Hence for the cycle we have an homology

$$\Gamma_k \sim \sum \gamma_\mu^i \times \delta_{k-\mu}^j + \sum \gamma_\mu^i \times b_{k-\mu}^j + \sum a_\mu^i \times \delta_{k-\mu}^j .$$

We may assume that the second sum cannot be split into two, one of which is a cycle, for it would then be of the same type as the first and could be merged with it. Similarly for the third sum. From this follows that if the second sum is absent so is the third and conversely. Also since the boundaries of the terms in the last two sums must cancel each other, any term $\gamma_\mu^i \times b_{k-\mu}^j$ in the second sum contains only such a γ as may come from the boundaries of terms in the third. γ_μ^i bounds then a certain $C_{\mu+1}$ of C_p. Let

$$b_{k-\mu}^j \equiv \sum \epsilon_{js} \delta_{k-\mu-1}^s .$$

At once,

$$C_{\mu+1} \times b_{k-\mu}^j \equiv \gamma_\mu^i \times b_{k-\mu}^j + (-1)^{\mu+1} \sum \epsilon_{js} C_{\mu+1} \times \delta_{k-\mu-1}^s \sim 0 \pmod{C_p \times C_q}.$$

On subtracting this bounding cycle from the second expression for Γ_k the term $\gamma_\mu^i \times b_{k-\mu}^j$ will have been replaced by a sum of terms that will go

* Coll. Lect., p. 116. The result in question, not stated explicitly by him, is really derived in the course of the discussion which leads to the theorem there stated.

into the first or the third sum. Therefore by a previous remark, the third must then also disappear. Hence

$$\Gamma_k \sim \sum \gamma_\mu^i \times \delta_{k-\mu}^i .$$

If γ_μ^i bounds $C_{\mu+1}$ of C_p, then $\gamma_\mu^i \times \delta_{k-\mu}^j$ bounds $C_{\mu+1} \times \delta_{k-\mu}^j$; hence in the expression of Γ_k only the terms for which neither γ nor δ bound need to be preserved, which proves our theorem.

It is not difficult to show that *if two factor cycles do not bound their product does not bound*, but as this is unnecessary for the sequel we omit the proof. A corollary is that *no cycle of the fundamental sets obtained bounds on* $C_p \times C_q$.

Of more import is the following observation. If γ_μ is a zero-divisor for C_p, $\gamma_\mu \times \delta_{k-\mu}$ is a zero-divisor or else bounds and similarly with δ in place of γ. For if $t\gamma_\mu$ bounds $C_{\mu+1}$, $(t\gamma_\mu) \times \delta_{k-\mu}$ bounds $C_{\mu+1} \times \delta_{k-\mu}$. Hence *the theorem that we have proved holds even when fundamental sets with respect to the operation* $\approx$ *take the place of the others.* For we may now eliminate from the fundamental sets of the product all cycles of which a factor is a zero-divisor.*

53. **Product of orientable manifolds.** We know already (No. 49) that if M_p, M_q are orientable manifolds so is $M_p \times M_q$. We are now especially concerned with the question of Kronecker indices.

Let E_h, E_{p-h} be simplicial cells on M_p and suppose that they intersect at A. Let E_k', E_{q-k}' and B be similar elements for M_q. Then $E_h \times E_k'$ and $E_{p-h} \times E_{q-k}'$ intersect at $A \times B$ on $M_p \times M_q$ and the question is *to determine their index in terms of* $(E_h \cdot E_{p-h})$, *and* $(E_k' \cdot E_{q-k}')$.

Let the A's and B's of No. 48 now serve to determine indicatrices for the manifolds, as previously for the cells. As a matter of fact, the M's might simply be E's without changing anything.

We may now assume that the indicatrices are

$$AA_1 \cdot \cdot \cdot A_h , \quad AA_{h+1} \cdot \cdot \cdot A_p \text{ for } E_h \text{ and } E_{p-h} ,$$

$$BB_1 \cdot \cdot \cdot B_k , \quad BB_{k+1} \cdot \cdot \cdot B_q \text{ for } E_k \text{ and } E_{q-k} .$$

Hence for the above named two-cell products the indicatrices

$$A \times B \; A_1 \times B \cdot \cdot \cdot A_h \times B \; A \times B_1 \cdot \cdot \cdot A \times B_k ,$$

$$A \times B \; A_{h+1} \times B \cdot \cdot \cdot A_p \times B \; A \times B_{k+1} \cdot \cdot \cdot A \times B_q .$$

* From this follow the results of my Transactions paper and those of Künneth, concerning connectivity and torsion indices.

By comparing with the indicatrix of $M_p \times M_q$ and applying the rule for the determination of the index we obtain the following formula:

$$(53.1) \quad (E_h \times E_k' \cdot E_{p-h} \times E_{q-k}') = (-1)^{k(p-h)} (E_h \cdot E_{p-h})(E_k' \cdot E_{q-k}') .$$

It holds even for $k=0$, $h=p$, when it yields the following result, verifiable directly with ease:

$$(53.2) \qquad (E_p \times B \cdot A \times E_q) = +1 .$$

The product $E_p \times B$ simply denotes an E_p on the product cell, and similarly for $A \times E_q$.

Let now C_h, C_{p-h} be complexes on M_p whose points of intersection are neither on their boundaries nor on that of the manifold, and let C_k', C_{q-k}' be analogous for M_q. Polyhedral approximations to C_h and C_k' have for product such an approximation to $C_h \times C_k'$, and similarly for the other two complexes. If the approximations to C_h and C_{p-h} and those to C_k' and C_{q-k}' intersect at isolated ordinary points of the complexes, the approximations to the products will behave likewise. Hence from the definition of the index of two polyhedral complexes with well defined intersections, the extension to arbitrary complexes, and (53.1), (53.2), we have

$$(53.3) \quad (C_h \times C_k' \cdot C_{p-h} \times C_{q-k}') = (-1)^{k(p-h)} (C_h \cdot C_{p-h})(C_k' \cdot C_{q-k}') ;$$

$$(53.4) \qquad (M_p \times B \cdot A \times M_q) = +1 .$$

The products in (53.4) represent complexes homeomorphic to M_p or M_q on $M_p \times M_q$. The approximations to A and B in that case consist in choosing points of their vicinity situated on p- and q-cells of the covering complexes of the manifolds which serve to define straightness and distances on them.

Two complexes C_h, C_{p-h-i} on M_p, without points on its boundary, may always be approximated by two that do not intersect. Hence if C_k', C_{q-k+i}' are on M_q, without points on its boundary, there are non-intersecting polyhedral approximations to $C_h \times C_k'$ and $C_{p-h-i} \times C_{q-k+i}'$. Therefore

$$(53.5) \qquad (C_h \times C_k' \cdot C_{p-h-i} \times C_{q-k+i}) = 0 .$$

54. THEOREM. *Let M_p, M_q be without boundary and let γ_k^i, δ_k^j be the cycles of their canonical fundamental sets. Then except for the signs their products constitute such a set for $M_p \times M_q$.*

This follows at once from the relations derived from (53.3), (53.4) (53.5):

$$(54.1)\quad (\gamma_\lambda^i \times \delta_{k-\lambda}^j \cdot \gamma_{p-\lambda}^i \times \delta_{q-k+\lambda}^j) = (-1)^{(k-\lambda)(p-\lambda)} \cdot (\gamma_\lambda^i \cdot \gamma_{p-\lambda}^i)(\delta_{k-\lambda}^j \cdot \delta_{q-k+\lambda}^j) ;$$

$$(54.2)\quad (\gamma_\lambda^i \times \delta_{k-\lambda}^j \cdot \gamma_{p-\mu}^i \times \delta_{q-k+\mu}^j) = 0 , \qquad \lambda \neq \mu .$$

They indicate also the manner in which the cycles are to be associated.

Of great importance for transformations is the special case $p=q=k=n$. We have then

$$(54.3)\quad (\gamma_\lambda^i \times \delta_{n-\lambda}^j \cdot \gamma_{n-\lambda}^i \times \delta_\lambda^j) = (-1)^{n(\lambda+1)} \cdot (\gamma_\lambda^i \cdot \gamma_{n-\lambda}^i) \cdot (\delta_\lambda^j \cdot \delta_{n-\lambda}^j) ,$$

and all other indices vanish.

55. Formulas (53.3), (53.4) are special cases of a more general one corresponding to cycles that are intersections of complexes, which is derived by similar considerations. As we shall not use it later we merely give it here without proof. If C_l and C_λ of M_p do not intersect each other's boundary, and have no intersection on the boundary of the manifold, if also C_m', C_μ' behave likewise relatively to M_q, the formula in question is as follows:

$$C_l \times C_m' \cdot C_\lambda \times C_\mu' \sim (-1)^{(p-l)(q-\mu)} \cdot C_l \cdot C_\lambda \times C_m' \cdot C_\mu' \qquad (\text{mod } M_p \times M_q) .$$

§2. Transformations of a manifold without boundary

56. Let M_n be the manifold, M_n' another copy of it, T a transformation of M_n into itself or part of itself, which we subject to this sole condition of a very general nature: *If A is any point of M_n, B the image of any transform A' of it on M_n', the set of all points $A \times B$ is an n-cycle Γ_n on $M_n \times M_n'$.* The inclusiveness of the class of transformations so defined becomes apparent when we remark that all continuous one-valued transformations (for each A only one A' varying continuously with A) belong to it. The non-singular C_n without boundary of which Γ_n is then the image in the sense of No. 6 is M_n itself. More generally k-valued continuous transformations are also of our type; the corresponding C_n is then kM_n.

Much of the rest of this paper will center around the determination of certain Kronecker indices and it becomes essential to define all orientations involved. Let C_n cover M_n and let C_n' be its image covering M_n'. Suppose that Π_n is a polyhedron which associated with C_n serves to define straightness and distances on M_n. The polyhedron Π_n has the same cell structure as C_n' and we shall agree to use it, associated with C_n', to define straightness

and distances on M_n'. Then any rectilinear segment of M_n has for image a similar one on M_n' and both have the same length. To an indicatrix E_n on M_n will correspond an E_n' on M_n'. We shall name the vertices of E_n' in the same order as the corresponding vertices of E_n, and use the simplex so sensed as indicatrix for M_n'. In accordance with our previous conventions the orientation of $M_n \times M_n'$ is now perfectly determined. Owing to its importance for the sequel, it is well perhaps to characterize it more geometrically. Through any point $A \times B$ of the manifold there pass $M_n \times B$ and $A \times M_n'$. Let B be the image of A. To E_n with A as its first vertex corresponds E_n' with B as its first vertex. Let $\bar{E}_n$ be the image of the first on $M_n \times B$, $\bar{E}_n'$ that of the second on $A \times M_n'$. The E_{2n} indicatrix of the product manifold has its vertices named in the following order: $A \times B$, the other vertices of $\bar{E}_n$, the other vertices of $\bar{E}_n'$.

There remains the orienting of Γ_n. If one cycle is suitable for T so is its opposite. Of the two we shall select the one such that

$$(\Gamma_n \cdot A \times M_n') = a_0 \geqq 0 ,$$

a perfectly definite condition since the integer is a Kronecker index of cycles.

57. Let us apply to M_n and M_n' the very notations of No. 54, so that δ_k^i is now the cycle of M_n' that corresponds to γ_k^i on M_n. We shall then have

$$\Gamma_n \sim \sum \epsilon_\mu^{ij} \cdot \gamma_\mu^i \times \delta_{n-\mu}^j \qquad (\text{mod } M_n \times M_n') . \tag{57.1}$$

The γ's and δ's are, it will be recalled, cycles of fundamental sets. In particular γ_0, δ_0 are merely points of M_n, M_n' while γ_n, δ_n are the manifolds themselves. The number of cycles of the fundamental sets is *one* for these two extreme cases.

The ϵ's are important characteristic integers of T. It will be remembered that two transformations are said to be of the same *class* if they belong to one and the same continuous family of transformations. Whenever T varies within its class, Γ_n is continuously deformed (a condition that might serve to define the class) and Γ_n remains homologous to a fixed cycle, so that the ϵ's are unchanged. Hence *to a given class of transformations corresponds a fixed set of ϵ's*. Whether the converse holds or not is as yet unknown.

58. We must now introduce the notion of the *transform of a cycle* of M_n by T. The necessity of defining such a notion with precision arises from the fact that while T is a point transformation a cycle is not a mere point set, but consists of a set of cells each taken with a certain multiplicity. The cycle is then really a symbol attached to a set of cells, and what is meant by its transform is by no means evident a priori.

Let γ_μ, $0 \leqq \mu \leqq n$, be any cycle of M_n. If it is not polyhedral we approximate it by one that is and that furthermore has been reduced in accordance with the theorem of No. 23. Then $\gamma_\mu \times M_n'$ behaves likewise and we may also approximate Γ_n in the same manner so that the two have a well defined intersection $\Gamma_n \,.\, \gamma_\mu \times M_n' = \Gamma_\mu$. (We economize in notations by designating the approximations like the cycles themselves.) To every point $A \times B$ of Γ_μ corresponds a unique B on M_n' varying continuously when $A \times B$ so varies on Γ_μ. Hence B gives rise to a cycle $\bar{\delta}_\mu$ on M_n'. It is a singular complex of M_n', continuous image of Γ_μ in the sense of No. 6. The image $\bar{\gamma}_\mu$ of $\bar{\delta}_\mu$ on M_n is by definition the transform of γ_μ by T. It will be observed that $\bar{\delta}_\mu$ is *polyhedral*, hence so is $\bar{\gamma}_\mu$. For $\bar{\delta}_\mu$ is represented on the copy $A \times M_n'$ of M_n' by the cycle $A \times \bar{\delta}_\mu$ which is the locus of all intersections of $A \times M_n'$ with all manifolds $M_n \times B$ that meet $\Gamma_n \,.\, \gamma_\mu \times M_n'$. In this fashion, however, $A \times \bar{\delta}_\mu$ appears as an intersection of polyhedral complexes, and it is therefore also polyhedral. If $A \times B \; A_1 \times B_1 \cdots A_\mu \times B_\mu$ is an indicatrix of Γ_μ then $BB_1 \cdots B_\mu$ is an indicatrix of $\bar{\delta}_\mu$.

The preceding definition is justified by the fact that for $\mu = 0$, when γ_0 consists of a finite number of points, the correct transforms of these points are obtained. Furthermore (No. 62) when T is the identity all cycles are left invariant, which is as it should be with any properly selected definition.

Let the cycle transformations be represented by homologies

$$\bar{\gamma}_\mu^i \sim \sum \alpha_\mu^{ij} \gamma_\mu^j \pmod{M_n} . \tag{58.1}$$

For any given T one is far more likely to possess information concerning the α's than concerning the ϵ's. Hence the interesting and important problem arises *to determine the ϵ's in terms of the α's*. This problem shall be solved partly in a more narrow form. Let us drop zero divisors throughout. In accordance with No. 52, with the R's denoting the connectivity indices of M_n, and assuming that among the cycles γ_μ^i the first R_μ are independent, we shall now have in place of (57.1) and (58.1), the following relations whose appearance is the same:

$$\Gamma_h \approx \sum \epsilon_\mu^{ij} \cdot \gamma_\mu^i \times \delta_{n-\mu}^j \pmod{M_n \times M_n'} ; \tag{58.2}$$

$$\bar{\gamma}_\mu^i \approx \sum \alpha_\mu^{ij} \gamma_\mu^j \pmod{M_n} , \tag{58.3}$$

where as before μ runs from 0 to n, but, for each μ, i and j run from 1 to $R_\mu = R_{n-\mu}$. The problem that we shall completely solve is *the determination of the ϵ's within this range in terms of the α's similarly limited*. This will suffice to provide all that is needed in our applications to fixed points and coincidences.

59. **An important formula.** The solution of our problem is based upon the following formula, whose proof will now occupy us:

$$(\Gamma_n \cdot \gamma_\mu \times \delta_{n-\mu}) = (-1)^\mu \cdot (\bar{\gamma}_\mu \cdot \gamma_{n-\mu}), \tag{59.1}$$

with indices computed at the left as to $M_n \times M_n'$ and at the right as to M_n. In place of (59.1) it will be found more convenient to prove the equivalent

$$(\Gamma_n \cdot \gamma_\mu \times \delta_{n-\mu}) = (-1)^\mu \cdot (\bar{\delta}_\mu \cdot \delta_{n-\mu}) , \tag{59.2}$$

where the last index is computed as to M_n', and to this we now turn our attention.

60. By means of our usual approximations as applied to Γ_n, γ_μ and $\delta_{n-\mu}$, we may so arrange matters that all cycles are polyhedral and have well defined intersections. Let $A \times B$ be an intersection of Γ_n with $\gamma_\mu \times \delta_{n-\mu}$. Then B is an intersection of $\bar{\delta}_\mu$ with $\delta_{n-\mu}$. Therefore we need only to show that the contributions of the two points to their respective indices have the ratio $(-1)^\mu$. Owing to the distributive law we may assume that all complexes are cells and that the points count for ± 1 in the indices, or what is the same, that the manifolds are linear spaces and T a projectivity. The method of matrices (No. 4) will be found most convenient for our purpose.

61. We begin with *the contribution of* $A \times B$ *to the left side of* (59.2). Let A, B, and $A \times B$ be origins of cartesian axes for the spaces M_n, M_n', and their product, the coördinates being $x_1, \cdots, x_n$ for M_n, $y_1, \cdots, y_n$ for M_n', $x_1, \cdots, x_n, y_1, \cdots, y_n$ for $M_n \times M_n'$. To the points (x), (y) of M_n, M_n' corresponds the point (x, y) of the product.

We can assume for M_n a matrix-indicatrix in the sense of No. 4, of type

$$\begin{Vmatrix} X_\mu & , & 0 \\ 0 & , & X_{n-\mu} \end{Vmatrix} \tag{61.1}$$

where the X's denote square matrices of order equal to the index with determinants equal to $+1$, and the zeros matrices whose terms all vanish. The meaning of (61.1) is clear. If A_i is the point of M_n whose coördinates are the terms in the ith row, then $AA_1 \cdots A_n$ is an indicatrix of M_n. We may so select it that $AA_1 \cdots A_\mu$ and $AA_{\mu+1} \cdots A_n$ are indicatrices of γ_μ and $\gamma_{n-\mu}$. This implies a definite orientation for them, but if we invert γ_μ or $\gamma_{n-\mu}$ we also invert $\bar{\delta}_\mu$ or $\delta_{n-\mu}$ and therefore (59.2) is unaffected, so that there is really no restriction. In the same sense as for M_n, we may say that the matrices

$$||X_\mu, 0|| \ , \ ||0, X_{n-\mu}||$$

define indicatrices of γ_μ and $\gamma_{n-\mu}$.

To the images of the transforms of $M_n, \gamma_\mu, \gamma_{n-\mu}$, by T, on M_n', correspond matrix-indicatrices

$$\left\| \begin{matrix} Y_\mu , Y \\ Y' , Y_{n-\mu} \end{matrix} \right\| , \quad || \; Y_\mu , Y \; || , \quad || \, Y' , Y_{n-\mu} \, ||$$

where Y, Y' are rectangular arrays and the two other Y's square arrays of order equal to their index. The first corresponds to $\pm M_n' = \epsilon M_n'$, where the sign is plus if T maintains the indicatrix on M_n, minus if it inverts it. The second and third matrices correspond to $\bar{\delta}_\mu$ and $\delta_{n-\mu}$. Since (61.1) defines an indicatrix of M_n' when each row is considered as the coördinates of a y point, and since its determinant, product of $|X_\mu|$ and $|X_{n-\mu}|$, is $+1$, ϵ has the sign of the determinant

$$\left| \begin{matrix} Y_\mu & , & Y \\ Y' & , & Y_{n-\mu} \end{matrix} \right| .$$

From the definition of Γ_n we conclude that

$$\left\| \begin{matrix} X_\mu , & 0 & , & Y_\mu & , & Y \\ 0 , & X_{n-\mu} , & & Y' & , & Y_{n-\mu} \end{matrix} \right\|$$

in a matrix-indicatrix of $\theta\Gamma_n$, $\theta = \pm 1$. To determine the sign of θ observe that since $A \times M_n'$ has for matrix-indicatrix

$$\left\| \begin{matrix} 0 & , & 0 & , & X_\mu & , & 0 \\ 0 & , & 0 & , & 0 & , & X_{n-\mu} \end{matrix} \right\| ,$$

the contribution of $A \times B$ to $(\theta\Gamma_n \cdot A \times M_n')$ has the sign of the determinant

$$\left| \begin{matrix} X_\mu & , & 0 & , & 0 & , & 0 \\ 0 & , & X_{n-\mu} & , & 0 & , & 0 \\ 0 & , & 0 & , & X_\mu & , & 0 \\ 0 & , & 0 & , & 0 & , & X_{n-\mu} \end{matrix} \right| = +1 .$$

Hence θ represents the contribution of the point to the integer $\alpha_0 = (\Gamma_n \cdot A \times M_n')$ of No. 56. For example, in the case of the identical transformation, every point of intersection of the two cycles must contribute $+1$, therefore $\theta = +1$. This is also true for a continuous 1 to n transformation, where the contributions to the index have a constant sign. That θ is not -1 in these two cases comes from the condition $\alpha_0 > 0$ by which we have definitely oriented Γ_n.

But this is a digression from our main topic to which we now return. It is readily seen that

$$\left\| \begin{matrix} X_\mu & , & 0 & , & 0 & , & 0 \\ 0 & , & 0 & , & 0 & , & X_{n-\mu} \end{matrix} \right\|$$

is a matrix-indicatrix for $\gamma_\mu \times \delta_{n-\mu}$. As $M_n \times M_n'$ has for matrix-indicatrix

$$\left\| \begin{matrix} X_\mu & , & 0 & , & 0 & , & 0 \\ 0 & , & X_{n-\mu} & , & 0 & , & 0 \\ 0 & , & 0 & , & X_\mu & , & 0 \\ 0 & , & 0 & , & 0 & , & X_{n-\mu} \end{matrix} \right\|$$

whose determinant is $+1$, we conclude from the definition of the Kronecker index that the contribution of $A \times B$ to $(\theta\Gamma_n \cdot \gamma_\mu \times \delta_{n-\mu})$ has the sign of the determinant

$$\left| \begin{matrix} X_\mu & , & 0 & , & Y_\mu & , & Y \\ 0 & , & X_{n-\mu} & , & Y' & , & Y_{n-\mu} \\ X_\mu & , & 0 & , & 0 & , & 0 \\ 0 & , & 0 & , & 0 & , & X_{n-\mu} \end{matrix} \right| = (-1)^\mu |Y_\mu| \, .$$

The contribution of $A \times B$ to $(\Gamma_n \cdot \gamma_\mu \times \delta_{n-\mu})$ has then the sign of $(-1)^n \theta |Y_\mu|$.

62. Let us now examine *the contribution of B to* $(\bar{\delta}_\mu \cdot \delta_{n-\mu})$. Suppose that we have found for $\Gamma_n \cdot \gamma_\mu \times M_n'$ a matrix-indicatrix

$$\| Z, Z' \|,$$

where Z, Z' are matrices with μ rows and n columns. Then Z' will define an indicatrix for $\bar{\delta}_\mu$, referred of course to the y coördinates. For any row $x_1, \cdots, x_n, y_1, \cdots, y_n$ represents a point $A' \times B'$ such that B' is the point of M_n' with the y coördinates. From this and the remark, No. 58, as to relations between indicatrices on Γ_n and $\bar{\delta}_\mu$, follows the property of Z'.

Now $\gamma_\mu \times (\epsilon M_n')$ has for matrix-indicatrix

$$\left\| \begin{matrix} X_\mu & , & 0 & , & 0 & , & 0 \\ 0 & , & 0 & , & Y_\mu & , & Y \\ 0 & , & 0 & , & Y' & , & Y_{n-\mu} \end{matrix} \right\|$$

which we may replace by

$$\left\| \begin{matrix} X_\mu & , & 0 & , & Y_\mu & , & Y \\ 0 & , & 0 & , & Y_\mu & , & Y \\ 0 & , & 0 & , & Y' & , & Y_{n-\mu} \end{matrix} \right\|,$$

derived from it by addition of rows, as this change of indicatrix merely corresponds to an affine transformation of determinant $+1$ applied to the $S_{\mu+n}$ involved, and is therefore permissible. The first row defines now an indicatrix of $\zeta\theta\Gamma_n \cdot \gamma_\mu \times (\epsilon M_n')$, where $\zeta = \pm 1$. From our mode of defining sensed intersections (No. 12) it follows that ζ has the same sign as the determinant

$$\begin{vmatrix} X_\mu & , & 0 & , & Y_\mu & , & Y \\ 0 & , & X_{n-\mu} & , & Y' & , & Y_{n-\mu} \\ 0 & , & 0 & , & Y_\mu & , & Y \\ 0 & , & 0 & , & Y' & , & Y_{n-\mu} \end{vmatrix} = \begin{vmatrix} Y_\mu & , & Y \\ Y' & , & Y_{n-\mu} \end{vmatrix},$$

that is, the sign of ϵ. As both are unity in absolute value, $\zeta = \epsilon$, so that actually the matrix

$$|| \; X_\mu \quad , \quad 0 \quad , \quad Y_\mu \quad , \quad Y \; ||$$

defines an indicatrix of $\epsilon\,\theta\Gamma_n \cdot \gamma_\mu \times (\epsilon M_n') = \theta\Gamma_n \cdot \gamma_\mu \times M_n'$. According to what has been said at the beginning of this number, then

$$|| \; Y_\mu \quad , \quad Y \; ||$$

defines an indicatrix for $\theta\bar{\delta}_\mu$.

Since M_n' has a matrix-indicatrix of determinant $+1$, the contribution of B to $(\theta\bar{\delta}_\mu \cdot \delta_{n-\mu})$ has the sign of the determinant

$$\begin{vmatrix} Y_\mu & , & Y \\ 0 & , & X_{n-\mu} \end{vmatrix},$$

since the last line is a matrix-indicatrix for $\delta_{n-\mu}$. As the determinant has for value $|Y_\mu|$, the contribution of B to $(\bar{\delta}_\mu \cdot \delta_{n-\mu})$ has the same sign as $\theta|Y_\mu|$. Its ratio to that of $(\Gamma_n \cdot \gamma_\mu \times \delta_{n-\mu})$ is then $(-1)^\mu$, which suffices to prove (59.1).

Remark. One incidental result follows readily from our discussion. The indicatrix of $\bar{\delta}_\mu$ coincides with the image of the transform of that of γ_μ or with its opposite according as θ is positive or negative. Upon translating this back to M_n itself, we find the following result. *If $A \times B$ contributes $+1$ to $(\Gamma_n \cdot A \times M_n')$ then the indicatrix of the transform of any cycle through A is the transform of the indicatrix; it is the opposite of it if the contribution is -1.* For example in the case of a continuous sense preserving 1 to n transformation, the indicatrix of the transform is always the transform of the indicatrix. In particular for the identical transformation the two coincide, and $\bar{\gamma}_\mu$ not only coincides with γ_μ element for element, but also

with preservation of orientation. This justifies in a sense our convention as to sensing Γ_n and our definition of the transform of a cycle (No. 58).

63. It is important to emphasize the fact that the extreme values $\mu=0$, n are not exceptional at all. For $\mu=0$, we have already imposed

$$(\Gamma_n \cdot A \times M_n') = a_0 . \tag{63.1}$$

The interpretation given for this integer in No. 60 fits in perfectly with our discussion. Let a_n be the similar integer for T^{-1}. Its interpretation is then this:* An arbitrary E_n of M_n is covered with a certain number of cells of $T \cdot M_n$. Among these there will be, say, k' positive cells, k'' negative cells of M_n and $a_n = k' - k''$. Here of course a_n may well be negative. Thus if T is a homeomorphism inverting the indicatrix, $a_n = -1$. An example of this is the symmetry of a sphere with respect to a diametral plane. It is important to remember that the T considered is not the initial transformation but one that corresponds to a polyhedral approximation of its Γ_n.

If we compute the index as to $M_n' \times M_n$ we obtain then

$$(\Gamma_n \cdot B \times M_n) = (\Gamma_n \cdot M_n \times B) = a_n .$$

If a manifold is inverted all corresponding indices must be changed in sign, an immediate corollary of their definition. As $M_n \times M_n' = (-1)^n M_n' \times M_n$, when the index is computed with reference to $M_n \times M_n'$,

$$(\Gamma_n \cdot M_n \times B) = (-1)^n a_n . \tag{63.2}$$

This is what (59.1) becomes for $\mu = n$. Both (63.1) and (63.2) can also be derived with ease by means of matrix-indicatrices. Indeed this would merely involve repeating the discussion of Nos. 60, 61.

Observe that the notation a_0, a_n is in accord with the meaning given to a_μ^{ij} in No. 57. They correspond to the homologies describing the behavior of the zero-cycle (a point) and the n-cycle (M_n itself). Explicitly for $\mu=0$, n, i and j can only be unity, and $a_0^{11} = a_0$, $a_n^{11} = a_n$.

64. We now pass to the actual determination of the ϵ's. If we remember that in calculating Kronecker indices zero-divisors may be dropped, we see that in applying (59.1) we may substitute from formulas (58.2) and (58.3) in place of (57.1) and (58.1). Let us then substitute in (59.1) for Γ_n its expression as given by (58.2), for γ_μ, $\delta_{n-\mu}$ the cycles γ_μ^h, $\delta_{n-\mu}^k$, a nd for $\bar{\gamma}_\mu^h$ its expression (58.3). We obtain

$$\sum_{i,j=1}^{R_\mu} \epsilon_{n-\mu}^{ij} (\gamma_{n-\mu}^{\ i} \times \delta_\mu^j \cdot \gamma_\mu^h \times \delta_{n-\mu}^k) = (-1)^\mu \sum_{j=1}^{R_\mu} a_\mu^{hj} (\gamma_\mu^j \cdot \gamma_{n-\mu}^{\ k}) ,$$

* This is the same as Brouwer's *degree* of T. See Mathematische Annalen, vol. 71 (1911), p. 105. He limits himself to the case where every point has a unique transform, which is a special case of transformations with $a_0 = 1$.

all other terms having dropped out in accordance with (54.1). Transforming the left side by means of (53.3) and (53.4) this becomes

$$(-1)^{\mu(n+1)} \sum_{i,j=1}^{R_\mu} \epsilon^{ij}_{n-\mu}(\gamma^k_\mu \cdot \gamma^j_{n-\mu})\,(\gamma^j_\mu \cdot \gamma^k_{n-\mu}) = \sum_{j=1}^{R_\mu} a^{kj}_\mu (\gamma^j_\mu \cdot \gamma^k_{n-\mu}) .$$

For each μ we have here a set of R_μ^2 linear equations in the B_μ unknowns $\epsilon^{ij}_{n-\mu}$. The determinant of their coefficients is a power of the determinant

$$\left|(\gamma^i_\mu \cdot \gamma^j_{n-\mu})\right| ,$$

whose value is actually ± 1, as may be deduced from Veblen's theorem mentioned in Part I, §6. These equations may therefore be solved even explicitly if need be, a task which presents little interest. However, if, *as we shall assume from now on, the fundamental sets on M_n are canonical the solution can be carried through in an instant.*

For if we substitute this time the indices as given in Part I, §6, and as they must be applied here, we have at once

(*a*) *$\mu \neq \frac{1}{2}n$ or $= \frac{1}{2}n =$ an even integer.*

Then

$$\epsilon^{ik}_{n-\mu} = a^{ik}_\mu , \qquad \mu > \tfrac{1}{2}n ;$$

$$\epsilon^{ik}_{n-\mu} = (-1)^{\mu(n+1)}\, a^{ik}_\mu , \qquad \mu \leqq \tfrac{1}{2}n .$$

(*b*) *$\mu = \frac{1}{2}n =$ an odd integer.*

Then

$$\epsilon^{2h-1,k}_{n/2} = -a^{2h,k}_{n/2} , \qquad \epsilon^{2h,k}_{n/2} = a^{2h-1,k}_{n/2} .$$

We obtain for Γ_n the following relations:

(*a*) $n \not\equiv 4$, mod 2.

$$\Gamma_n \approx \sum_{0 \leq \mu < n/2} \sum_{i,j=1}^{R_\mu} a^{ij}_{n-\mu} \cdot \gamma^i_\mu \times \delta^j_{n-\mu}$$

$$+ \sum_{\mu \geqq n/2} (-1)^{(n+1)\mu} \sum_{i,j=1}^{R_\mu} a^{ij}_{n-\mu} \cdot \gamma^i_\mu \times \delta^j_{n-\mu} .$$

(*b*) $n \equiv 4$, mod 2.

$$\Gamma_n \approx \sum_{0 \leq \mu < n/2} \sum_{i,j=1}^{R_\mu} (a^{ij}_{n-\mu} \cdot \gamma^i_\mu \times \delta^j_{n-\mu} + (-1)^\mu a^{ij}_\mu \cdot \gamma^i_{n-\mu} \times \delta^j_\mu)$$

$$+ \sum_{h=1}^{\frac{1}{2}R_{n/2}} \sum_{k=1}^{R_{n/2}} (a^{2h-1}_{n/2} \cdot \gamma^{2h}_{n/2} \times \delta^k_{n/2} - a^{2h,k}_{n/2} \cdot \gamma^{2h-1}_{n/2} \times \delta^k_{n/2}) .$$

65. **A new proof of Veblen's theorem on fundamental sets.** We refer to the theorem of Part I, §6, according to which the invariant factors of the matrix

$$|| (\gamma_\mu^i \cdot \gamma_{n-\mu}^j) ||$$

for any two fundamental sets are all unity. Let them be in any case e_1, $e_2, \cdots, e_{R_\mu}$. That their number is R_μ follows from other considerations that need not detain us.* The reduction to canonical sets will be accomplished as before, only now

$$(\gamma_\mu^i \cdot \gamma_{n-\mu}^i) = \pm e_i ,$$

the same indices being zero as previously. Therefore we have now

$$\epsilon_{n-\mu}^{hk} \cdot e_h e_k = \pm a_\mu^{hk} \cdot e_k .$$

Hence e_h is a factor of a_μ^{hk}, and in particular of a_μ^{hh} *for every* T. But for the identical transformation $a_\mu^{hh} = 1$, hence $e_h = 1$ which is precisely Veblen's theorem.

§3. Coincidences and fixed points of transformations of a manifold

66. A *coincidence* of two transformations T, T' is a pair of points A, A', of M_n, such that A' is a transform of A, by both T and T'. A *fixed point* of T is a coincidence for T and the identical transformation. Let Γ_n, Γ_n' be the cycles corresponding to T and T', and let B be the image of A' on M_n'; $A \times B$ is an intersection of Γ_n with Γ_n' and conversely to such an intersection corresponds a coincidence of T and T'. The determination of the *number of coincidences and fixed points* is then reduced to a question of intersections of cycles. The actual numbers are not definite, may even vary for transformations of the same class, become infinite, and so on. Not so, however, with the attached Kronecker index $(\Gamma_n \cdot \Gamma_n')$, whose determination alone is usually possible. Its interpretation is simple enough. If we consider suitable approximations to T and T' there will be only a finite number of intersections. Some of these, say k', shall be counted positively, others, say k'', counted negatively according to a definite rule, and the difference $k' - k''$ is independent of the mode of approximation, in a sense sufficiently clear in the light of Part I, §5. From what has been established there, it follows that *the number of coincidences to be obtained in the sequence is an actual topological invariant of the transformations involved*, the same being

* See my Monograph, p. 13; also Veblen's paper on this question quoted in Part I, §6.

true for *the number of fixed points* counted in an analogous manner to the coincidences.

67. We shall now endeavor to characterize topologically the coincidences according to the signs of their contributions to $(\Gamma_n \cdot \Gamma_n')$. We select two polyhedral approximations that intersect in a finite number of points only, in such a way that if $A \times B$ is one of them it has neighborhoods on the cycles and on $M_n \times M_n'$ that are interiors of simplexes. We continue to call the approximations Γ_n, Γ_n'. Each intersection $A \times B$ counts for ± 1 in computing the index, and as far as its neighborhood alone and those of A and B on M_n and M_n' are concerned, everything is as if all complexes involved were linear spaces. The matrix-indicatrix method will then be again the most convenient.

To Γ_n and Γ_n' we may make correspond matrices

$$|| \; X_n \; , \; Y_n \; || \; , \; || \; X_n' \; , \; Y_n' \; || \; .$$

Regarding X_n and X_n' we may replace them by any others whose determinants have the same signs. This is an immediate corollary of the definition of the indicatrix. Let us assume our coördinate system such that the unit matrix of order n, I_n, corresponds to an indicatrix for M_n as referred to the x coördinates. Then I_n will play the same part for M_n' and the y coördinates, I_{2n} for $M_n \times M_n'$ and the coördinates x, y. Under the circumstances, as we have shown in No. 60, if θ, θ' denote the contributions of $A \times B$ to the indices $(\Gamma_n \cdot A \times M_n')$, $(\Gamma_n' \cdot A \times M_n')$, then the determinants in question have the sign of θ and θ'. We conclude that there will be suitable indicatrices for $\theta\Gamma_n$ and $\theta'\Gamma_n'$ of type

$$|| \; I_n \; , \; Y_n \; || \quad , \quad || \; I_n \; , \; Y_n' \; || \; .$$

Now the contribution of $A \times B$ to $(\theta\Gamma_n \cdot \theta'\Gamma_n')$, in absolute value equal to unity, has the sign of

$$\begin{vmatrix} I_n \; , & Y_n \\ I_n \; , & Y_n' \end{vmatrix},$$

since I_{2n} is a matrix-indicatrix for $M_n \times M_n'$. This determinant is equal to $|Y_n' - Y_n|$, and therefore the contribution of $A \times B$ to $(\Gamma_n \cdot \Gamma_n')$ has the sign of $\theta\theta' |Y_n' - Y_n|$. The determinant factor is certainly $\neq 0$, else $A \times B$ would not be an isolated intersection of the Γ's. Furthermore owing to the degree of arbitrariness in the choice of these cycles we may always assume that the equation of the nth degree

$$f(t) = \; | \; Y_n' - tY_n \; | = 0$$

(characteristic equation of $Y_n'-tY_n$) has only distinct roots. We know already that $f(1)\neq 0$.

Let $e=\pm 1$ be the same integer as in No. 60 ($+1$ if T maintains an indicatrix of M_n at A, -1 if it inverts it). Then as shown there

$$\epsilon\,|\,Y_n\,|>0\,.$$

Since the determinant Y_n is the leading coefficient of $f(t)$, $f(1)$ has the sign of $(-1)^\nu\epsilon$, where ν is the number of real roots of $f(t)$ that are less than unity. Therefore the contribution of $A\times B$ to $(\Gamma_n\cdot\Gamma_n')$ *is equal to* $(-1)^\nu\epsilon\theta\theta'$.

68. It is decidedly desirable and worth while to find a geometric interpretation for ν. The transformation T acts as an affine transformation whereby the vector V whose x components are $\xi_1, \xi_2, \cdots, \xi_n$ is changed into a vector whose components are defined by the matrix product.

$$||\,\xi_1\,,\,\xi_2\,,\cdots,\,\xi_n\,||\cdot Y_n\,,$$

and similarly for T' and Y_n'. Let t_i be a real root of $f(t)$. Then there is a V_i whose transforms by T and T' are collinear and in the ratio $1:t_i$. Let $V_1, V_2, \cdots, V_\nu$ be the similar vectors for all real roots <1. The vectors

$$u_1V_2+\cdots+u_\nu V_\nu\,;\qquad 0\leqq u_i\leqq 1\,,\ \sum u_i=1$$

fill up a simplex Σ_ν^0 of M_n with a vertex at A. Its transforms by T and T' are simplexes Σ_ν, Σ_ν' with the common vertex A', the second being entirely on the first, with all cells through A' situated in the same linear spaces, and ν is the largest integer for which such simplexes exist. This is the desired interpretation for ν.

69. We now proceed with the determination of $(\Gamma_n\cdot\Gamma_n')$.

Referring to No. 54, and recalling the special relations for canonical sets, we find that the only indices $(\gamma_\lambda^i\times\bar\delta_{n-\lambda}^j\cdot\gamma_\mu^k\times\delta_{n-\mu}^k)\neq 0$ are those for which $\lambda+\mu=n$ and whose values are given below.

(*a*) $\mu\neq\frac{1}{2}n$ *or else* $=\frac{1}{2}n$ *an even integer.*

$$\begin{aligned}(\gamma_\mu^i\times\delta_{n-\mu}^j\cdot\gamma_{n-\mu}^i\times\delta_\mu^j)&=(-1)^{n(\mu+1)}(\gamma_\mu^i\cdot\gamma_{n-\mu}^i)(\gamma_\mu^j\cdot\gamma_{n-\mu}^j)\\&=(-1)^{n(\mu+1)};\quad 1\leqq i\,,j\leqq R_\mu\,,\end{aligned}$$

for with canonical systems, the two indices dropped are equal, their common value being ± 1.

(*b*) $\mu=\frac{1}{2}n$ *an odd integer.* For entirely similar reasons the indices below alone are left, with values as given:

$$(\gamma^{2k-1}\times\delta^{2k-1}\cdot\gamma^{2k}\times\delta^{2k})=(\gamma^{2k}\times\delta^{2k}\cdot\gamma^{2k-1}\times\delta^{2k-1})=-1\,,$$

$$(\gamma^{2k-1}\times\delta^{2k}\cdot\gamma^{2k}\times\delta^{2k-1})=(\gamma^{2k}\times\delta^{2k-1}\cdot\gamma^{2k-1}\times\delta^{2k})=+1\,.$$

To facilitate the reading of these formulas we have omitted the lower indices whose common value is $\frac{1}{2}n$.

70. The application to $(\Gamma_n \cdot \Gamma_n')$ is immediate. Let, throughout, the η's and β's play the same part for Γ_n' or T' as the ϵ's and α's for Γ_n or T. The distributive law for indices gives

$$(\Gamma_n \cdot \Gamma_n') = \sum_{i,j,\mu} \epsilon_{n-\mu}^{ij}\, \eta_\mu^{hk}\ (\gamma_\mu^i \sim \delta_{n-\mu}^j \cdot \gamma_{n-\mu}^h \times \delta_\mu^k)\ .$$

Therefore for $n \not\equiv 2$, mod 4, and with μ replaced by $n-\mu$,

$$(70.1) \qquad (\Gamma_n \cdot \Gamma_n') = \sum_{\mu=0}^{n} (-1)^{n\mu} \sum_{i,j=1}^{R\mu} \epsilon_{n-\mu}^{ij}\, \eta_\mu^{ij}\ ,$$

while for $n \equiv 2$, mod 4, after some simplifications,

(70.2) $(\Gamma_n \cdot \Gamma_n')$ = (same sum as in (70.1) except that μ does not take the value $\frac{1}{2}n$)

$$+ \sum_{h,k=1}^{\frac{1}{2}R_{n/2}} \left(\epsilon_{n/2}^{2h-1,2k}\, \eta_{n/2}^{2h,2k-1} + \epsilon_{n/2}^{2h,2k-1}\, \eta_{n/2}^{2h-1,2k} - \epsilon_{n/2}^{2h-1,2k-1}\, \eta_{n/2}^{2h,2k} - \epsilon_{n/2}^{2h,2k}\, \eta_{n/2}^{2h-1,2h-1} \right) .$$

In terms of the α's and β's, by means of No. 64 we find, if $n \not\equiv 2$, mod 4,

$$(70.3) \qquad (\Gamma_n \cdot \Gamma_n') = \sum_{\mu=0}^{n} (-1)^{\mu} \sum_{i,j=1}^{R\mu} \alpha_\mu^{ij}\, \beta_{n-\mu}^{ij}\ ,$$

and if $n \equiv 2$, mod 4,

(70.4) $(\Gamma_n \cdot \Gamma_n')$ = (same sum as in (70.3) except that μ does not take the value $\frac{1}{2}n$)

$$+ \sum_{h,k=1}^{\frac{1}{2}R_{n/2}} \left(\alpha_{n/2}^{2h-1,2k}\, \beta_{n/2}^{2h,2k-1} + \alpha_{n/2}^{2h,2k-1}\, \beta_{n/2}^{2h-1,2k} - \alpha_{n/2}^{2h-1,2k-1}\, \beta_{n/2}^{2h,2k} - \alpha_{n/2}^{2h,2k}\, \beta_{n/2}^{2h-1,2k-1} \right) .$$

71. To obtain the number of invariant points of T, always counted with a certain sign, it is convenient to replace throughout T by the identical transformation, whose cycle we denote by Γ_n^0, and T' by T. The homologies for the identical transformation are (No. 62)

$$\bar{\gamma}_\mu^h \sim \gamma_\mu^h\ ,$$

hence $\alpha_\mu^{ii}=1$, $\alpha_\mu^{ij}=0$ if $i\neq j$. On replacing the α's by these values and the β's by the α's both (70.3) and (70.4) reduce to the very simple formula

$$(71.1)\qquad (\Gamma_n^0\cdot\Gamma_n)=\sum_{\mu=0}^{n}(-1)^\mu\sum_{i=1}^{R_\mu}\alpha_\mu^{ii}\,.$$

Remarkably enough *this formula is still correct if the fundamental sets cease to be canonical.* For if $\gamma_\mu'^i$ is a new fundamental set, we have

$$\gamma_\mu'^i\approx\sum_j A_{ij}\gamma_\mu^j \qquad (\text{mod } M_n)\,,$$

where the A's are integers whose determinant

$$|A_{ij}|=\pm 1\,.$$

Hence the transformation matrix of the γ''s that takes the place of

$$||\alpha_\mu^{ij}||$$

is obtained by transforming the latter by means of the matrix of the A's, an operation which leaves unchanged the sum of the terms in the principal diagonal, which is the term corresponding to μ in (71.1).

A particularly noteworthy case is when the effect of T on any cycle is merely to increase it by a zero-divisor, which includes as a special case deformations. Then *all* the α's in (71.1) are equal to $+1$ and

$$(\Gamma_n\cdot\Gamma_n)=\sum_{\mu=0}^{n}(-1)^\mu R_\mu\,.$$

This expression is the well known Euler-Poincaré *characteristic number of* M_n (difference between the number of cells of even dimensionality of any covering C_n and the number of the rest). Since $R_\mu=R_{n-\mu}$ its value is zero for n odd, hence this very neat proposition: *for a T of the preceding type, in particular of the same class as the identity, the number of invariant points counted with their signs is equal to the Euler-Poincaré characteristic. It is zero when n is odd.*

Thus for $n=2$ the number is $2-2p$ as found by Birkhoff, who however confined himself to analytic transformations.

Fixed points for transformations of hyperspheres have proved of importance in many questions. There are then no cycles of dimensions other than 0 or n so that (71.1) becomes

$$(\Gamma_n^0\cdot\Gamma_n)=\alpha_0+(-1)^n\alpha_n\,,$$

a result obtained by Brouwer for $\alpha_0=1$.

Finally formula (71.1) for $n=2$ includes practically all related results obtained in recent years by various authors (Brouwer, Nielsen, Kérékjárto, Alexander).

72. Another class of applications is to *coincidences of algebraic correspondences on algebraic curves*. Owing to a proposition which I proved in my Monograph, p. 19, the Kronecker indices give then the *exact* number of intersections. We should therefore expect to be able to identify (70.4) with the well known coincidence formula due to A. Hurwitz.* Observe that his h'_{ki}, etc., are the same as the h_{ki}, etc., not for C' but for C'^{-1} and should be replaced as indicated in his footnote; for example, h'_{ik} by G'_{ik} and so on. With these changes and a comparison of notations the identification becomes complete.

Thus *the coincidence formulas may be obtained without making use of any function theory*. The classical Hurwitz relations between the periods are simply the analytical translation of the transformations on the linear cycles of M_n.

* Mathematische Annalen, vol. 28 (1887), p. 578, formula 35. See in this connection Chisini, Istituto Lombardo Rendiconti, ser. 2, vol. 7 (1924), p. 481.

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DIVERGENT DOUBLE SEQUENCES AND SERIES*

BY

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I. INTRODUCTION

Several definitions for giving a value to a divergent simple series, as for example the Cesàro's and Hölder's means, can be expressed by means of a linear transformation defined by an infinite matrix on numbers. Two types of these transformations are given as follows, one by a triangular matrix, the other by a square matrix.

$$T: \quad \begin{vmatrix} a_{1,1} & & & & \\ a_{2,1} & a_{2,2} & & & \\ a_{3,1} & a_{3,2} & a_{3,3} & & \\ a_{4,1} & a_{4,2} & a_{4,3} & a_{4,4} & \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{vmatrix}$$

$$S: \quad \begin{vmatrix} a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} & a_{1,5} & \cdot & \cdot & \cdot & \cdot \\ a_{2,1} & a_{2,2} & a_{2,3} & a_{2,4} & a_{2,5} & \cdot & \cdot & \cdot & \cdot \\ a_{3,1} & a_{3,2} & a_{3,3} & a_{3,4} & a_{3,5} & & & & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{vmatrix}$$

For any given sequence $\{x_n\}$ a new sequence $\{y_n\}$ is defined as follows:

$$y_n = \sum_{k=1}^{n} a_{n,k}\, x_k, \text{ for the matrix } T\ ,$$

$$y_n = \sum_{k=1}^{\infty} a_{n,k}\, x_k, \text{ for the matrix } S\ ,$$

provided in the latter case y_n has a meaning. If to any matrix of type T we adjoin the elements $a_{n,k}=0$, $k>n$ (all n), we obtain a matrix of type S. Since this addition does not affect the transformation, any transformation of the type T may be considered as a special case of a transformation of type S. If for either transformation $\lim_{n\to\infty} y_n$ exists, the limit is called the generalized value of the sequence x_n by the transformation. If whenever x_n converges, y_n converges to the same value, then the transformation

* Presented to the Society, December 29, 1920; received by the editors in March, 1922.

is said to be *regular*. The criterion for regularity of these transformations is stated as follows:

THEOREM.* *A necessary and sufficient condition that the transformation T be regular is that*

(a) $\lim_{n\to\infty} a_{n,k}=0$ *for every* k,

(b) $\lim_{n\to\infty} \sum_{k=1}^{n} a_{n,k}=1$,

(c) $\sum_{k=1}^{n} |a_{n,k}|<A$ *for all* n ,

(a) $\lim_{n\to\infty} a_{n,k}=0$ *for each* k,

THEOREM.† *A necessary and sufficient condition that the transformation S be regular is that*

(b) $\sum_{k=1}^{\infty} |a_{n,k}|$ *converge for each* n,

(c) $\sum_{k=1}^{\infty} |a_{n,k}|<A$ *for all* n,

(d) $\lim_{n\to\infty} \sum_{k=1}^{\infty} a_{n,k}=1$.

Corresponding to these definitions of summability for a single series, we have the following definitions for giving a value to a divergent double series. Let the given series be represented as follows:

$$\begin{aligned} &u_{1,1}+u_{1,2}+u_{1,3}+u_{1,4}+u_{1,5}+\cdots \\ &+u_{2,1}+u_{2,2}+u_{2,3}+u_{2,4}+u_{2,5}+\cdots \\ &+u_{3,1}+u_{3,2}+u_{3,3}+u_{3,4}+\cdots \\ &+u\cdots\cdots\cdots

Thus we have also

$$u_{m,n}=x_{m,n}+x_{m-1,n-1}-x_{m,n-1}-x_{m,1-n},\ m>1,\ n>1\ ;$$
$$u_{1,n}=x_{1,n}-x_{1,n-1n},\ n>1\ ;$$
$$u_{m,1}=x_{m,1}-x_{m-1,1},\ m>1\ ;$$
$$u_{1,1}=x_{1,1}.$$

We define a new sequence by the relation

$$y_{m,n}=\sum_{k=1,l=1}^{m,n} a_{m,n,k,1}\,x_{k,1l}\ ,$$

We shall call this transformation and its matrix $A:(a_{m,n,k,l})$ of the type T; here $k\leqq m$; $l\leqq n$. Again we may write

$$y_{m,n}=\sum_{k=1,l=1}^{\infty,\infty} a_{m,n,k,1}\,x_{k,1l}\ .$$

provided $y_{m,n}$ has a meaning. We shall call this transformation and its matrix $A:(a_{m,n,k,l})$ of the type S; here k and l take on all positive integral values. Any transformation of type T may be considered as a special case of a transformation of type S; for by adding the elements

$$a_{m,n,k,l}=0,\ m<k,\ n<l,\ \text{all } m \text{ and } n,$$
$$a_{m,n,k,l}=0,\ 1\leqq k\leqq m,\ n<l,\ \text{all } m \text{ and } n,$$
$$a_{m,n,k,l}=0,\ m<k,\ 1\leqq l\leqq n,\ \text{all } m \text{ and } n,$$

to any matrix of the type T we obtain a matrix of type S such that the resulting transformation is identical with the original one. If for either transformation $y_{m,n}$ possesses a limit, the limit is called the generalized value of the sequence $x_{m,n}$ by the transformation.

It is a well known fact that if a simple series converges, the corresponding sequence in bounded. This need not hold for a double series. Thus consider the series

$$u_{1,n}=1,$$
$$u_{2,n}=-1,$$
$$u_{m,n}=0,\ m\geqq 3.$$

This series converges, but the corresponding sequence is not bounded. Thus convergent double series may be divided into two classes according to whether the corresponding sequences are bounded or not. The following definition for regularity of a transformation is constructed with regard to a convergent bounded sequence; thus even if a transformation is regular

it need not give to an unbounded convergent sequence the value to which it converges.

If whenever $x_{m,n}$ is a bounded convergent sequence, $y_{m,n}$ converges to the same value, then the transformation is said to be regular. A regular transformation of real elements $(a_{m,n,k,l})$ is said to be totally regular, provided when applied to a sequence of real elements $(x_{m,n})$, which has the following properties,

(*a*) $x_{m,n}$ is bounded for each m,

(*b*) $x_{m,n}$ is bounded for each n,

(*c*) $\lim_{m,n\to\infty} x_{m,n}=+\infty$,

it transforms this sequence into a sequence which has for its limit $+\infty$.

Concerning these transformations we shall prove necessary and sufficient conditions for regularity, and then for total regularity. We shall give also conditions which these definitions must satisfy in order that $\lim_{m,n\to\infty} y_{m,n}$ shall exist, whenever $\lim_{m,n\to\infty} x_{m,n}$ exists, irrespective of their values; furthermore for the case where the sequence $x_{m,n}$ is merely bounded.

II. Regularity of linear transformations

The criterion for regularity of the transformations of type T is given in the following theorem:

Theorem I. *A necessary and sufficient condition that any transformation of type T be regular is*

(*a*) $\lim_{m,n\to\infty} a_{m,n,k,l}=0$, *for each k and l,*

(*b*) $\lim_{m,n\to\infty} \sum_{k=1,l=1}^{m,n} a_{m,n,k,l}=1$,

(*c*) $\lim_{m,n\to\infty} \sum_{k=1}^{m} |a_{m,n,k,l}|=0$, *for each l,*

(*d*) $\lim_{m,n\to\infty} \sum_{l=1}^{n} |a_{m,n,k,l}|=0$, *for each k,*

(*e*) $\sum_{k=1,l=1}^{m,n} |a_{m,n,k,l}|\leqq A$,

where A is some constant.

Proof of necessity. (*a*) Define a sequence $(x_{m,n})$ as follows: $x_{m,n}=1$, $m=p$, $n=q$; $x_{m,n}=0$, except when $m=p$, $n=q$. Then $\lim_{m,n\to\infty} x_{m,n}=0$, $y_{m,n}=a_{m,n,p,q}$.

Hence in order that $\lim_{m,n\to\infty} y_{m,n}=0$, it is necessary that $\lim_{m,n\to\infty} a_{m,n,p,q}=0$ for each p and q. Thus condition (a) is necessary.

(b) Consider the sequence $(x_{m,n})$ defined as follows: $x_{m,n}=1$. Then $y_{m,n}=\sum_{k=1,l=1}^{m,n} a_{m,n,k,l}$. Since $\lim_{m,n\to\infty} y_{m,n}=1$, condition (b) is necessary.

(c) To show the necessity of condition (c) we assume that condition (a) is satisfied and that (c) is not, and obtain a contradiction. Since we are assuming that for $l=l_0$ (some fixed integer) the sequence $\sum_{k=1}^{m}|a_{m,n,k,l}|$ does not approach zero, for some preassigned constant $h>0$ there must exist a sub-sequence of this sequence, such that each element of it is greater than h.

Choose m_1 and n_1 such that

$$\sum_{k=1}^{m_1} |a_{m_1,n_1,k,l_0}|>h.$$

Choose $m_2>m_1$; $n_2>n_1$ and such that

$$\sum_{k=1}^{m_1} |a_{m_2,n_2,k,l_0}|\leqq\frac{h}{2},\quad \sum_{k=1}^{m_2} |a_{m_2,n_2,k,l_0}|>h,$$

and in general choose $m_p>m_{p-1}$; $n_p>n_{p-1}$ and such that

$$\sum_{k=1}^{m_{p-1}} |a_{m_p,n_p,k,l_0}|<\frac{h}{2^{p-1}},\quad \sum_{k=1}^{m_p} |a_{m_p,n_p,k,l_0}|>h. \tag{1}$$

From (1) we have

$$\sum_{k=m_{p-1}+1}^{m_p} |a_{m_p,n_p,k,l_0}|>h-\frac{h}{2^{p-1}}=h\left(1-\frac{1}{2^{p-1}}\right). \tag{2}$$

Define a sequence $(x_{m,n})$ as follows:

$$\begin{aligned}
&x_{m,n}=0, \quad n\neq l_0 ;\\
&x_{m,n}=\operatorname{sgn} a_{m_1,n_1,k,l_0}, \quad m\leqq m_1 ;\\
&x_{m,n}=\operatorname{sgn} a_{m_2,n_2,k,l_0}, \quad m_1<m\leqq m_2 ;\\
&\qquad\cdots\cdots\cdots\\
&x_{m,n}=\operatorname{sgn} a_{m_p,n_p,k,l_0}, \quad m_{p-1}<m\leqq m_p ;\\
&\qquad\cdots\cdots\cdots.
\end{aligned} \tag{3}$$

Here $\lim_{m,n\to\infty} x_{m,n}=0$. For this $(x_{m,n})$ sequence we have

$$\begin{aligned}
y_{m_p,n_p}&=\sum_{k=1}^{m_p} a_{m_p,n_p,k,l_0}\,x_{k,l_0}\\
&=\sum_{k=1}^{m_{p-1}} a_{m_p,n_p,k,l_0}\,x_{k,l_0}+\sum_{k=m_{p-1}+1}^{m_p} a_{m_p,n_p,k,l_0}\,x_{k,l_0}.
\end{aligned}$$

From (1), (2), and (3) it follows that

$$\left| \sum_{k=1}^{m_{p-1}} a_{m_p,n_p,k,l_0} x_{k,l_0} \right| \leqq \sum_{k=1}^{m_{p-1}} \left| a_{m_p,n_p,k,l_0} \right| \leqq \frac{h}{2^{p-1}},$$

$$\sum_{k=m_{p-1}+1}^{m_p} a_{m_p,n_p,k,l_0} x_{k_1,l_0} = \sum_{k=m_{p-1}+1}^{m_p} \left| a_{m_p,n_p,k,l_0} \right| \geqq h\left(1-\frac{I}{2^{p-1}}\right).$$

Hence

$$y_{m_p,n_p} \geqq h\left(1-\frac{1}{2^{p-1}}\right) - \frac{1}{2^{p-1}} = h\left(1-\frac{1}{2^{p-2}}\right).$$

Thus $y_{m,n}$ does not have the limit zero, from which follows the necessity of condition (c).

(d) The above proof can be used for showing the necessity of condition (d) by simply interchanging the rôles of rows and columns.

(e) Assume conditions (a) and (b) are satisfied and that (e) does not hold. Choose m_1 and n_1 such that

$$\sum_{k=1,l=1}^{m_1,n_1} |a_{m_1,n_1,k_1,l}| \geqq 1.$$

Choose $m_2 > m_1$, $n_2 > n_1$, and such that

$$\sum_{k=1,l=1}^{m_1,n_1} |a_{m_2,n_2,k,l}| \leqq 2,$$

$$\sum_{k=1,l=1}^{m_2,n_2} |a_{m_2,n_2,k,l}| \geqq 2^4,$$

and, in general, choose $m_p > m_{p-1}$, $n_p > n_{p-1}$ and such that

$$\text{(4)} \qquad \begin{aligned} \sum_{k=1,l=1}^{m_{p-1},n_{p-1}} |a_{m_p,n_p,k,l}| &\leqq 2^{p-1}, \\ \sum_{k=1,l=1}^{m_p,n_p} |a_{m_p,n_p,k,l}| &\geqq 2^{2p}. \end{aligned}$$

From equations (4) we have

$$\sum_{k=1,l=n_{p-1}+1}^{m_{p-1},n_p} |a_{m,n,k,l}| + \sum_{k=m_{p-1}+1,\ l=1}^{m_p,n_{p-1}} |a_{m_p,n_p,k,l}|$$

$$+ \sum_{k=m_{p-1}+1,\ l=n_{p-1}+1}^{m_p,n_p} |a_{m_p,n_p,k,l}| \geqq 2^{2p} - 2^{p-1} \geqq 2^{2p} - 2^{2p-1} = 2^{2p-1}.$$

We now have two sequences of integers

$$m_1 < m_2 < m_3 < m_4 \cdots,$$
$$n_1 < n_2 < n_3 < n_4 \cdots,$$

such that

$$\sum_{k=1,l=1}^{m_{p-1},n_{p-1}} |a_{m_p,n_p,k,l}| \leqq 2^{p-1},\ p>1,$$

$$\sum_{k=1,l=n_{p-1}+1}^{m_{p-1},n_p} |a_{m_p,n_p,k,l}| + \sum_{k=m_{p-1}+1,l=1}^{m_p,n_{p-1}} |a_{m_p,n_p,k,l}|$$

(5)

$$+ \sum_{k=m_{p-1}+1,l=n_{p-1}+1}^{m_p,n_p} |a_{m_p,n_p,k,l}| \leqq 2^{2p-1}.$$

Define a sequence $(x_{m,n})$ as follows:

$$x_{m,n} = \operatorname{sgn} a_{m_1,n_1,k,l},\ k \leqq m_1,\ l \leqq n_1,$$

$$x_{m,n} = \tfrac{1}{2} \operatorname{sgn} a_{m_2,n_2,k,l} \qquad \left\{\begin{matrix} m_1 < k \leqq m_2,\ 1 \leqq l \leqq n_1 \\ 1 \leqq k \leqq m_1,\ m_1 < l \leqq n_2 \\ m_1 < k \leqq m_2,\ n_1 < l \leqq n_2 \end{matrix}\right\};$$

(6)

$$\cdots\cdots\cdots$$

$$x_{m,n} = \frac{1}{2^{p-1}} \operatorname{sgn} a_{m_p,n_p,k,l} \qquad \left\{\begin{matrix} m_{p-1} < k \leqq m_p,\ 1 \leqq l = n_{p-1} \\ 1 \leqq k \leqq m_{p-1},\ n_{p-1} < l \leqq n_p \\ m_{p-1} < k \leqq m_p,\ n_{p-1} < l \leqq n_p \end{matrix}\right\}.$$

$$\cdots\cdots\cdots.$$

Here $\lim_{m,n\to\infty} x_{m,n} = 0$. Consider

$$y_{m_p,n_p} = \sum_{k=1,l=1}^{m_p,n_p} a_{m_p,n_p,k,l}\, x_{k,l} = \sum_{k=1,l=1}^{m_{p-1},n_{p-1}} a_{m_p,n_p,k,l}\, x_{k,l}$$

$$+ \sum_{k=1,l=n_{p-1}+1}^{m_{p-1},n_p} a_{m_p,n_p,k,l}\, x_{k,l} + \sum_{k=m_{p-1}+1,l=1}^{m_p,n_{p-1}} a_{m_p,n_p,k,l}\, x_{k,l}$$

$$+ \sum_{k=m_{p-1}+1,l=n_{p-1}+1}^{m_p,n_p} a_{m_p,n_p,k,l}\, x_{k,l}.$$

From (5) and (6) we have

$$\left| \sum_{k=1,l=1}^{m_{p-1},n_{p-1}} a_{m_p,n_p,k,l}\, x_{k,l} \right| \leqq \sum_{k=1,l=1}^{m_{p-1},n_{p-1}} \left| a_{m_p,n_p,k,l} \right| \leqq 2^{p-1},$$

$$\sum_{k=1,\,l=n_{p-1}+1}^{m_{p-1},n_p} a_{m_{p-1},n_p,k,l}x_{k,l}+\sum_{k=m_{p-1}+1,\,l=1}^{m_p,n_{p-1}} a_{m_p,n_p,k,l}x_{k,l}$$

$$+\sum_{k=m_{p-1}+1,\,l=n_{p-1}+1}^{m_p,n_p} a_{m_p,n_p,k,l}x_{k,l}=\frac{1}{2^{p-1}}\left[\sum_{k=1,\,l=n_{p-1}+1}^{m_{p-1},n_p}|a_{m_p,n_p,k,l}|\right.$$

$$\left.+\sum_{k=m_{p-1}+1,\,l=1}^{m_p,n_{p-1}}|a_{m_p,n_p,k,l}|+\sum_{k=m_{p-1}+1,\,l=1}^{m_p,n_p}|a_{m,n,k,l}|\right]$$

$$\geqq\frac{1}{2^{p-1}}2^{2p-1}=2^p\,.$$

Hence

$$|y_{m_p,n_p}|\geqq 2^p-2^{p-1}=2^{p-1}\,.$$

Thus

$$\lim_{p\to\infty}|y_{m_p,n_p}|=\infty\,.$$

Since this sub-sequence of the $(y_{m,n})$ sequence does not converge, the sequence $(y_{m,n})$ has no limit and thus condition (e) is necessary.

Proof of sufficiency. Let the limit of the convergent sequence $(x_{m,n})$ be x; then

$$y_{m,n}-x=\sum_{k=1,\,l=1}^{m,n}a_{m,n,k,l}x_{k,l}-x\,.$$

From condition (b) we may write

$$\sum_{k=1,\,l=1}^{m,n}a_{m,n,k,l}+r_{m,n}=1\,, \tag{7}$$

where

$$\lim_{m,n\to\infty}r_{m,n}=0\,.$$

Therefore

$$y_{m,n}-x=\sum_{k=1,\,l=1}^{m,n}a_{m,n,k,l}(x_{k,l}-x)-r_{m,n}x;$$

$$\begin{aligned}|y_{m,n}-x|\leqq&\left|\sum_{k=1,\,l=1}^{p,q}a_{m,n,k,l}(x_{k,l}-x)\right|+\left|\sum_{k=1,\,l=q+1}^{p,n}a_{m,n,k,l}(x_{k,l}-x)\right|\\&+\left|\sum_{k=p+1,\,l=1}^{m,q}a_{m,n,k,l}(x_{k,l}-x)\right|+\left|\sum_{k=p+1,\,l=q+1}^{m,n}a_{m,n,k,l}(x_{k,l}-x)\right|\\&+|r_{m,n}x|\,.\end{aligned} \tag{8}$$

Since $x_{m,n}\to x$, we can choose p and q so large that for any preassigned small constant ϵ

$$|x_{k,l}-x|<\frac{\epsilon}{5A}, \text{ whenever } k\geqq p,\ l\geqq q\,.$$

Let L be the greatest of the numbers $|x_{k,l}-x|$ for all k and l. Now choose M and N such that whenever $m\geqq M$, $n\geqq N$, the following inequalities are satisfied:

(i) $$\sum_{k=1,l=1}^{p,q}|a_{m,n,k,l}|<\frac{\epsilon}{5pqL}$$ (from condition (a))

(ii) $$|r_{m,n}|<\frac{\epsilon}{5|x|}$$ (from equation (7)),

(iii) $$\sum_{k=1}^{m}|a_{m,n,k,l}|<\frac{\epsilon}{5qL}\,,\ l=1,2,\cdots,q$$ (from condition (c)),

(iv) $$\sum_{l=1}^{n}|a_{m,n,k,l}|<\frac{\epsilon}{5pL}\,,\ k=1,2,\cdots,p$$ (from condition (d)).

Hence whenever $m\geqq M$, $n\geqq N$ we have

$$|y_{m,n}-x|\leqq\frac{\epsilon}{5pqL}pqL+\frac{\epsilon}{5pL}pL+\frac{\epsilon}{5qL}qL+\frac{\epsilon}{5A}A+\frac{\epsilon}{5|x|}|x|=\epsilon\,.$$

Thus

$$\lim_{m,n\to\infty}y_{m,n}-x=0\,,$$

or $y_{m,n}\to x$, which proves the theorem.

The following examples show that neither condition (c) nor condition (d) follows from (e). We define a transformation of type T as follows:

$$a_{m,n,k,l}=\frac{1}{2^k n}\,.$$

Here

(a) $$\lim_{m,n\to\infty}a_{m,n,k,l}=0\,,$$

(b) $$\lim_{m,n\to\infty}\sum_{k=1,l=1}^{m,n}a_{m,n,k,l}=\lim_{m,n\to\infty}\sum_{k=1}^{n}\frac{1}{2k}=1\,,$$

(c) $$\lim_{m,n\to\infty}\sum_{k=1}^{m}a_{m,n,k,l}\leqq\lim_{m,n\to\infty}\frac{1}{n}=0\,,$$

$$(d) \qquad \lim_{m,n\to\infty} \sum_{l=1}^{n} a_{m,n,k,l} = \lim_{m,n\to\infty} \frac{1}{2^k} = \frac{1}{2^k} \neq 0 ,$$

$$(e) \qquad \sum_{k=1,l=1}^{m,n} |a_{m,n,k,l}| = \sum_{k=1,l=1}^{m,n} a_{m,n,k,l} \leqq 1 .$$

If we consider the transformation ($a_{m,n,k,l} = 1/2^l m$), we shall find conditions (a), (b), (c) and (e) satisfied, but not (d).

Let us further assume that the elements ($a_{m,n,k,l}$) of the transformation are real and positive. We now have

$$\sum_{k=1,l=1}^{m,n} |a_{m,n,k,l}| = \sum_{k=1,l=1}^{m,n} a_{m,n,k,l} .$$

It does not follow as in the case of the simple sequence that if $\lim_{m,n\to\infty} \sum_{k=1,l=1}^{m,n} a_{m,n,k,l} = 1$, then $\sum_{k=1,l=1}^{m,n} a_{m,n,k,l}$ is bounded for all values of m and n. Hence condition (e) of the preceding theorem does not follow from condition (b). Thus we see that the criterion for regularity of a transformation is not simplified by making the above assumption.

Furthermore we have

$$y_{m,n} = \sum_{k=1,l=1}^{m,n} a_{m,n,k,l}\, x_{k,l} .$$

Taking absolute values we obtain the following inequality:

$$|y_{m,n}| \leqq \sum_{k=1,l=1}^{m,n} |a_{m,n,k,l}\, x_{k,l}| \leqq \sum_{k=1,l=1}^{m,n} |a_{m,n,k,l}|\ K \leqq A\ K ,$$

where $|x_{k,l}| \leqq K$. Hence the

COROLLARY. *Any bounded sequence* ($x_{m,n}$) *is transformed by a regular transformation of type T into a bounded sequence* ($y_{m,n}$).

Before considering the regularity of a transformation of type S, we will prove certain lemmas.

LEMMA I. *If* $a_{m,n} \geqq 0$, $\sum_{m=1,n=1}^{\infty,\infty} a_{m,n}$ *diverges, it is possible to find a bounded sequence* $\epsilon_{m,n}$ *such that*

(i) $\epsilon_{m,n} = 0$,

(ii) $\lim_{m,n\to\infty} \epsilon_{m,n} = 0$,

(iii) $\sum_{k=1,l=1}^{\infty} a_{k,l}\epsilon_{k,l}$ *diverges.*

Proof. Let $S_{m,n} = \sum_{k=1,l=1}^{m,n} a_{k,l}$. Therefore $S_{m+1,n+1} \geqq S_{m,n+1} \geqq S_{m,n}$, $S_{m+1,n+1} \geqq S_{m+1,n} \geqq S_{m,n}$. Thus $\lim_{m,n\to\infty} S_{m,n} = +\infty$.

We define

$$\epsilon_{m,n}=\begin{cases}\dfrac{1}{S_{m,n}}, & \text{if } S_{m,n}\neq 0,\\ 0, & \text{if } S_{m,n}=0.\end{cases}$$

Here

(i) $\epsilon_{m,n}$ is bounded and $\geqq 0$,

(ii) $\lim_{m,n\to\infty}\epsilon_{m,n}=0$.

From some value of m, n onward,

$$\epsilon_{m,n}\geqq\epsilon_{m+1,n}\geqq\epsilon_{m+1,n+1},\quad \epsilon_{m,n}\geqq\epsilon_{m,n+1}\geqq\epsilon_{m+1,n+1}.$$

For a fixed m, n, where $\epsilon_{m,n}\neq 0$, we have

$$\begin{aligned}&\sum_{k=m+1,l=n+1}^{p,q}a_{k,l}\epsilon_{k,l}+\sum_{k=1,l=n+1}^{m,q}a_{k,l}\epsilon_{k,l}+\sum_{k=m+1,l=1}^{p,n}a_{k,l}\epsilon_{k,l}\\ &\geqq\epsilon_{m+p,n+q}\left[\sum_{k=m+1,l=n+1}^{p,q}a_{k,l}+\sum_{k=1,l=n+1}^{m,q}a_{k,l}+\sum_{k=m+1,l=1}^{p,n}a_{k,l}\right]\\ &=\epsilon_{m+p,n+q}(S_{m+p,n+q}-S_{m,n}).\end{aligned}\tag{9}$$

In the above summation $\epsilon_{m+p,n+q}$ is in the smallest term except those which are zero, but these terms drop out of the summation. Now choose p and q so large that

$$S_{m,n}\leqq\frac{1}{2}S_{m+p,n+q}.$$

Then equation (9) becomes

$$\sum_{k=m+1,l=n+1}^{p,q}a_{k,l}\epsilon_{k,l}+\sum_{k=1,l=n+1}^{m,q}a_{k,l}\epsilon_{k,l}+\sum_{k=m+1,l=1}^{p,n}a_{k,l}\epsilon_{k,l}\geqq\frac{1}{2}\epsilon_{m+p,n+q}S_{m+p,n+q}=\frac{1}{2}.$$

Setting $m_1=p$, $n_1=q$, this process can be repeated. Since this can be carried out an infinite number of times, we have

$$\sum_{m=1,n=1}^{\infty,\infty}a_{m,n}\epsilon_{m,n}=\infty.$$

LEMMA II. If $\sum_{m=1,n=1}^{\infty,\infty}a_{m,n}$ *is not absolutely convergent, it is possible to choose a bounded sequence* $x_{m,n}$ *such that* $x_{m,n}\to 0$ *and* $\sum_{m=1,n=1}^{\infty,\infty}a_{m,n}x_{m,n}$ *diverges to* $+\infty$.

Proof. Under the hypothesis $\sum_{m=1,n=1}^{\infty,\infty} |a_{m,n}|$ diverges. Now choose $\epsilon_{m,n}$ as in the preceding lemma with regard to the series $\sum_{m=1,n=1}^{\infty,\infty} |a_{m,n}|$. We define $x_{m,n}=\epsilon_{m,n}$ sgn $a_{m,n}$; then $a_{m,n}x_{m,n}=a_{m,n}\epsilon_{m,n}$ sgn $a_{m,n}=\epsilon_{m,n}|a_{m,n}|$.

By the preceding lemma $\sum_{m=1,n=1}^{\infty,\infty} a_{m,n}x_{m,n}$ diverges, as we wished to prove.

We now proceed to consider transformations of type S.

THEOREM II. *In order that whenever a bounded sequence $(x_{m,n})$ possesses a limit x, $\sum_{k=1,l=1}^{\infty,\infty} a_{m,n,k,l}x_{k,l}$ shall converge and $\lim_{m,n\to\infty} \sum_{k=1,l=1}^{\infty,\infty} a_{m,n,k,l} \cdot x_{k,l}=x$, it is necessary and sufficient that*

(*a*) $\lim_{m,n\to\infty} a_{m,n,k,l}=0$ *for each k and l,*

(*b*) $\sum_{k=1,l=1}^{\infty,\infty} |a_{m,n,k,l}|$ *converge for each m and n,*

(*c*) $\lim_{m,n\to\infty} \sum_{k=1,l=1}^{\infty,\infty} a_{m,n,k,l}=1$,

(*d*) $\lim_{m,n\to\infty} \sum_{k=1}^{\infty} |a_{m,n,k,l}|=0$ *for each l,*

(*e*) $\lim_{m,n\to\infty} \sum_{l=1}^{\infty} |a_{m,n,k,l}|=0$ *for each k,*

(*f*) $\sum_{k=1,l=1}^{\infty,\infty} |a_{m,n,k,l}| \leqq A$ *for all m and n.*

Proof of necessity. (*a*) Define a sequence $(x_{m,n})$ as follows: $x_{m,n}=1$, $m=p$, $n=q$; $x_{m,n}=0$ except when $m=p$, $n=q$. Here $y_{m,n}=a_{m,n,k,q}$. Hence condition (*a*) is necessary.

(*b*) Choose any fixed m and n and assume $\sum_{k=1,l=1}^{\infty,\infty} |a_{m,n,k,l}|$ diverges; then there exists by the preceding Lemma II a bounded sequence $x_{m,n}$ having the limit zero, and such that $\sum_{k=1,l=1}^{\infty,\infty} a_{m,n,k,l}x_k$ diverges. This contradicts our hypothesis; hence condition (*b*) is necessary.

(*c*) Consider the sequence $(x_{m,n})$ defined as follows: $x_{m,n}=1$, all m and n; then $y_{m,n}=\sum_{k=1,l=1}^{\infty,\infty} a_{m,n,k,l}$; and thus condition (*c*) is necessary.

(*d*) To show the necessity of condition (*d*) we assume that conditions (*a*) and (*b*) are satisfied but that (*d*) does not hold; and obtain a contradiction. Since the double sequence $\sum_{k=1}^{\infty} |a_{m,n,k,l_0}|$ (l_0 being a fixed integer) does not approach zero, then for some preassigned small constant $h>0$ there must exist a sub-sequence of this sequence such that every element of it is greater than h.

Choose m_1, n_1, and r_1 at random.

Choose $m_2>m_1$, $n_2>n_1$ such that

$$\sum_{k=1}^{r_1} |a_{m_2,n_2,k,l_0}| \leqq \frac{h}{8}, \text{ from } (a),$$

$$\sum_{k=1}^{\infty} |a_{m_2,n_1,k,l_0}| \geqq h;$$

and $r_2>r_1$ such that

$$\sum_{k=r_2+1}^{\infty} |a_{m_2,n_2,k,l_0}| \leqq \frac{h}{8}, \text{ from } (b).$$

In general choose $m_p>m_{p-1}$, $n_p>n_{p-1}$ such that

$$\sum_{k=1}^{r_{p-1}} |a_{m_p,n_p,k,l_0}| \leqq \frac{h}{8}, \text{ from } (a),$$

$$\sum_{k=1}^{\infty} |a_{m_p,n_p,k,l_0}| \geqq h;$$

and $r_p>r_{p-1}$ such that

$$\sum_{k=r_{p-1}+1}^{\infty} |a_{m_p,n_p,k,l_0}| \geqq \frac{h}{8}, \text{ from } (b) \tag{11}$$

From (10) and (11) we have

$$\sum_{k=r_{p-1}+1}^{r_p} |a_{m_p,n_p,k,l_0}| \geqq \frac{3}{4}h. \tag{12}$$

Define a sequence $(x_{m,n})$ as follows:

$$x_{k,l}=0, \quad l\neq l_0;$$

$$x_{k,l}=\begin{cases} \operatorname{sgn} a_{m_1,n_1,k,l_0}, & k\leqq r_1, \\ \operatorname{sgn} a_{m_2,n_2,k,l_0}, & r_1<k\leqq r_2, \\ \cdots\cdots \\ \operatorname{sgn} a_{m_p,n_p,k,l_0}, & r_{p-1}<k\leqq r_p, \\ \cdots\cdots \end{cases} \tag{13}$$

Here $\lim_{m,n\to\infty} x_{k,l}=0$. From (10) and (11),

$$\left|\sum_{k=1}^{r_{p-1}} a_{m_p,n_p,k,l_0}\, x_{k,l_0}\right| \leqq \sum_{k=1}^{r_{p-1}} |a_{m_p,n_p,k,l_0}| \leqq \frac{h}{8}.$$

From (12) and (13),

$$\left|\sum_{k=r_{p-1}+1}^{r_p} a_{m_p,n_p,k,l_0}\, x_{k,l_0}\right| = \sum_{k=r_{p-1}+1}^{r_p} |a_{m_p,n_p,k,l_0}| \geqq \frac{3}{4}h.$$

Consider

$$|y_{m_p,n_p}| = \Big| \sum_{k=1}^{\infty} a_{m_p,n_p,k,l_0}\, x_{k,l_0} \Big|$$

$$\geqq \sum_{k=r_{p-1}+1}^{r_p} |a_{m_p,n_p,k,l_0}\, x_{k,l_0}| - \Big| \sum_{k=1}^{r_{p-1}} a_{m_p,n_p,k,l_0}\, x_{k,l_0} \Big|$$

$$- \Big| \sum_{k=r_p+1}^{\infty} a_{m_p,n_p,k,l_0}\, x_{k,l_0} \Big|$$

$$\geqq \frac{3}{4}h - \frac{2h}{8} = \frac{h}{2}\,.$$

Thus $\lim_{m,n\to\infty} y_{m,n}$ is not zero, hence (d) is necessary.

(e) In a similar manner we can show the necessity of (e).

(f) Assume conditions (a) and (b) are satisfied and that (f) does not hold. Choose any $m_1, n_1; r_1, s_1$ at random.

Choose $m_2 > m_1$; $n_2 > n_1$ such that

$$\sum_{k=1,l=1}^{m_1,n_1} |a_{m_2,n_2,k,l}| \leqq 2\,, \text{ from } (a)\,;$$

$$\sum_{k=1,l=1}^{m_2,n_2} |a_{m_2,n_2,k,l}| \geqq 2^4\,.$$

Choose $r_2 > r_1$; $s_2 > s_1$ such that

$$\sum_{k=r_2+1,l=s_2+1}^{\infty,\infty} |a_{m_2,n_2,k,l}| + \sum_{k=1,l=s_2+1}^{r_2,\infty} |a_{m_2,n_2,k,l}|$$

$$+ \sum_{k=r_2+1,l=1}^{\infty,s_2} |a_{m_2,n_2,k,l}| \leqq 2^2\,, \text{ from } (b)\,.$$

Choose $m_3 > m_2$; $n_3 > n_2$ such that

$$\sum_{k=1,l=1}^{m_2,n_2} |a_{m_3,n_3,k,l}| \leqq 2^2\,,$$

$$\sum_{k=1,l=1}^{\infty,\infty} |a_{m_3,n_3,k,l}| \geqq 2^6\,.$$

Choose $r_3 > r_2$, $s_3 > s_2$ such that

$$\sum_{k=r_2+1,l=s_2+1}^{\infty,\infty} |a_{m_3,n_3,k,l}| + \sum_{k=r_2+1,l=1}^{\infty,s_3} |a_{m_3,n_3,k,l}|$$

$$+ \sum_{k=1,l=s_2+1}^{r_2,\infty} |a_{m_3,n_3,k,l}| \leqq 2^4\,.$$

In general, choose $m_p > m_{p-1}$, $n_p > n_{p-1}$, such that

$$\text{(14)} \qquad \begin{aligned} &\sum_{k=1,l=1}^{m_{p-1},n_{p-1}} |a_{m_p,n_p,k,l}| \leqq 2^{p-1}\,, \\ &\sum_{k=1,l=1}^{\infty,\infty} |a_{m_p,n_p,k,l}| \geqq 2^{2p}\,; \end{aligned}$$

and $r_p > r_{p-1}$, $s_p > s_{p-1}$ such that

$$\text{(15)} \qquad \begin{aligned} &\sum_{k=r_{p-1}+1,l=s_{p-1}+1}^{\infty,\infty} |a_{m_p,n_p,k,l}| + \sum_{k=r_{p-1}+1,l=1}^{\infty,s_p} |a_{m_p,n_p,k,l}| \\ &\quad + \sum_{k=1,l=s_{p-1}+1}^{r_p,\infty} |a_{m_p,n_p,k,l}| \leqq 2^{2p-2}\,. \end{aligned}$$

From these inequalities we have

$$\text{(16)} \qquad \begin{aligned} &\sum_{k=r_{p-1}+1,l=s_{p-1}+1}^{r_p,s_p} |a_{m_p,n_p,k,l}| + \sum_{r=1,l=s_{p-1}+1}^{r_{p-1},s_p} |a_{m_p,n_p,k,l}| + \sum_{k=r_{p-1}+1,l=1}^{s_p,s_{p-1}} |a_{m_p,n_p,k,l}| \\ &\quad \geqq 2^{2p} - 2^{2p-2} - 2^{p-1} = 2^{p-1}\left[2^{p+1} - 2^{p-1} - 1\right] \\ &\quad \geqq 2^{p-1}\left[2^{p+1} - 2^{p-1} - 2^{p-1}\right] = 2^{2p-1}\,. \end{aligned}$$

Define a sequence $(x_{m,n})$ as follows:

$$\text{(17)} \qquad \begin{aligned} &x_{k,l} = \operatorname{sgn} a_{m_1,n_1,k,l},\ k \leqq m_1\,,\ l \leqq n_1\,; \\ &x_{k,l} = \frac{1}{2} \operatorname{sgn} a_{m_2,n_2,k,l} \qquad \left\{ \begin{array}{l} 1 \leqq k \leqq m_1,\ n_1 < l \leqq n_2 \\ m_1 < k \leqq m_2,\ 1 \leqq l \leqq n_1 \\ m_1 < k \leqq m_2,\ n_1 < l \leqq n_2 \end{array} \right\}; \\ &\qquad \cdots \qquad \cdots \qquad \cdots \qquad \cdots \\ &x_{k,l} = \frac{1}{2^{p-1}} \operatorname{sgn} a_{m_p,n_p,k,l} \qquad \left\{ \begin{array}{l} 1 \leqq k \leqq m_{p-1},\ n_{p-1} < l \leqq n_p \\ m_{p-1} < k \leqq m_p,\ 1 < l \leqq n_{p-1} \\ m_{p-1} < k \leqq m_p,\ n_{p-1} < l \leqq n_p \end{array} \right\} \end{aligned}$$

Here $\lim_{m,n\to\infty} x_{m,n} = 0$.

From the preceding inequalities (14), (15), (16) and (13) we have

$$\text{(18)} \qquad \left| \sum_{k=1,l=1}^{m_{p-1},n_{p-1}} a_{m_p,n_p,k,l}\, x_{k,l} \right| \leqq \sum_{k=1,l=1}^{m_{p-1},n_{p-1}} |a_{m_p,n_p,k,l}| \leqq 2^{p-1}\,;$$

$$\sum_{k=m_{p-1}+1,\,l=n_{p-1}+1}^{m_p,n_p} a_{m_p,n_p,k,l}\,x_{k,l}+\sum_{k=1,\,l=n_{p-1}+1}^{m_{p-1},n_p} a_{m_p,n_p,k,l}\,x_{k,l}$$

$$(19)\quad +\sum_{k=m_{p-1}+1,\,l=1}^{m_p,n_{p-1}} a_{m_p,n_p,k,l}\,x_{k,l}=\frac{1}{2^{p-1}}\Big[\sum_{k=m_{p-1}+1,\,l=n_{p-1}+1}^{m_p,n_p}|a_{m_p,n_p,k,l}|$$

$$+\sum_{k=1,\,l=n_{p-1}+1}^{m_{p-1},n_p}|a_{m_p,n_p,k,l}|+\sum_{k=m_{p-1}+1,\,l=1}^{m_p,n_p}|a_{m_p,n_p,k,l}|\Big]\geqq\frac{1}{2^{p-1}}\,2^{2p-1}=2^p\,;$$

$$\Big|\sum_{k=m_p+1,\,l=n_p+1}^{\infty,\infty} a_{m_p,n_p,k,l}\,x_{k,l}+\sum_{k=1,\,l=n_p+1}^{m_p,\infty} a_{m_p,n_p,k,l}\,x_{k,l}$$

$$(20)\quad +\sum_{k=m_p+1,\,l=1}^{\infty,n_p} a_{m_p,n_p,k,l}\,x_{k,l}\Big|\leqq\frac{1}{2^p}\Big[\sum_{k=m_p+1,\,l=n_p+1}^{\infty,\infty}|a_{m_p,n_p,k,l}|$$

$$+\sum_{k=1,\,l=n_p+1}^{m_p,\infty}|a_{m_p,n_p,k,l}|+\sum_{k=m_p+1,\,l=1}^{\infty,n_p}|a_{m_p,n_p,k,l}|\Big]\leqq\frac{1}{2^p}2^{2p-2}=2^{p-2}\,.$$

Consider

$$|y_{m_p,n_p}|=\Big|\sum_{k=1,\,l=1}^{\infty,\infty} a_{m_p,n_p,k,l}\,x_{k,l}\Big|$$

$$\geqq\Big|\sum_{k=m_{p-1}+1,\,l=n_{p-1}+1}^{m_p,n_p} a_{m_p,n_p,k,l}\,x_{k,l}+\sum_{k=1,\,l=n_{p-1}+1}^{m_{p-1},n_p} a_{m_p,n_p,k,l}\,x_{k,l}$$

$$+\sum_{k=m_{p-1}+1,\,l=1}^{m_p,n_{p-1}} a_{m_p,n_p,k,l}\,x_{k,l}\Big|-\Big|\sum_{k=m_p+1,\,l=n_p+1}^{\infty,\infty} a_{m_p,n_p,k,l}\,x_{k,l}$$

$$+\sum_{k=1,\,l=n_p+1}^{m_p,\infty} a_{m_p,n_p,k,l}x_{k,l}+\sum_{k=m_p+1,\,l=1}^{\infty,n_p} a_{m_p,n_p,k,l}\,x_{k,l}\Big|$$

$$-\Big|\sum_{k=1,\,l=1}^{m_p,n_p} a_{m_p,n_p,k,l}\,x_{k,l}\Big|\geqq 2^p-2^{p-1}-2^{p-2}=2^{p-2}[4-2-1]$$

$$=2^{p-2}\text{, from (18) , (19) , (20) .}$$

Hence $\lim_{p\to\infty}|y_{m_p,n_p}|=+\infty$. Thus the sequence $(y_{m,n})$ has no finite limit, hence (e) is necessary.

Proof of sufficiency. From definition we can write

$$y_{m,n}-x=\sum_{k=1,\,l=1}^{\infty,\infty} a_{m,n,k,l}\,x_{k,l}-x.$$

From condition (c) we have

$$\sum_{k=1,\,l=1}^{\infty,\infty} a_{m,n,k,l}+r_{m,n}=1,$$

where $\lim_{m,n\to\infty} r_{m,n}=0$. Hence

$$y_{m,n}-x=\sum_{k=1,l=1}^{\infty,\infty} a_{m,n,k,l}\,(x_{k,l}-x)+r_{m,n}x;$$

$$|y_{m,n}-x| \leqq \Big|\sum_{k=1,l=1}^{p,q} a_{m,n,k,l}\,x_{k,l}\Big| + \Big|\sum_{k=1,l=q+1}^{p,\infty} a_{m_p,n_p,k,l}\,x_{k,l}\Big|$$

$$+\Big|\sum_{k=p+1,l=1}^{\infty,q} a_{m,n,k,l}\,x_{k,l}\Big| + \Big|\sum_{k=p+1,l=q+1}^{\infty,\infty} a_{m_p,n_p,k,l}\,x_{k,l}\Big| + |r_{m,n}x|.$$

Given $\epsilon>0$ we can choose p and q so large that

$$|x_{k,l}-x| \leqq \frac{\epsilon}{5A}, \quad \text{when } k>p, \quad l>q.$$

Let L be the greatest of the numbers $|x_{k,l}-x|$ for all k and l. Using conditions (a), (d), and (e) we can choose two integers M and N such that whenever $m\geqq M$, $n\geqq N$,

(i) $$\sum_{k=1,l=1}^{p,q} |a_{m_p,n_p,k,l}| < \frac{\epsilon}{5pqL};$$

(ii) $$\sum_{l=1}^{\infty} |a_{m_p,n_p,k,l}| < \frac{\epsilon}{5pL} \qquad (k=1, 2, 3, \cdots, p);$$

(iii) $$\sum_{k=1}^{\infty} |a_{m_p,n_p,k,l}| < \frac{\epsilon}{5qL} \qquad (l=1, 2, 3, \cdots, q);$$

(iv) $$|r_{m,n}| < \frac{\epsilon}{5|x|}.$$

We thus have, whenever $m>M$, $n>N$,

$$|y_{m,n}-x| \leqq \frac{\epsilon}{5Lpq}\,Lpq + \frac{\epsilon}{5Lp}\,Lp + \frac{\epsilon}{5Lq}\,Lq$$

$$+\frac{\epsilon}{5|x|}|x| + \frac{\epsilon}{5A}\,A = \epsilon.$$

Hence

$$\lim_{m,n\to\infty} y_{m,n}=x.$$

Thus the theorem is proved.

From the equation

$$y_{m,n}=\sum_{k=1,l=1}^{\infty,\infty} a_{m,n,k,l}\,x_{k,l}$$

we have, taking absolute values,

$$|y_{m,n}| \leqq \sum_{k=1,l=1}^{\infty,\infty} a_{m,n,k,l}\, x_{k,l} \leqq \sum_{k=1,l=1}^{\infty,\infty} |a_{m,n,k,l}|\, K \leqq A\ K,$$

where $|x_{k,l}| \leqq K$. Hence the

COROLLARY. *A bounded sequence $(x_{m,n})$ is transformed by a regular transformation of type S into a bounded sequence $(y_{m,n})$.*

If in the proof of the preceding theorem we replace the set of integers 1, 2, 3, $\cdots$ as range of variation of m by a point set T having a limiting point t_0 as range of variation of a variable t and the set of integers 1, 2, 3, $\cdots$ as range of variations of n by a point set V having a limiting point v_0 as range of variation of a variable V, we shall have the proof of the following generalization of the theorem.

THEOREM III. *Let $a_{k,l}(t, v)$ be defined for $k=1, 2, 3, \cdots, l=1, 2, 3, \cdots$, t in T, v in V, where T and V are two point sets in the real or complex plane, having t_0 and v_0 (finite or infinite) respectively as limit points. Then in order that whenever the bounded sequence $(x_{m,n})$ possesses a finite limit x, it should be true that for each pair of values t and v,*

$$\sum_{k=1,l=1}^{\infty,\infty} a_{k,l}(t, v)\, x_{k,l} \quad \textit{exists}$$

and that

$$\lim_{t\to t_0, v\to v_0} \sum_{k=1,l=1}^{\infty,\infty} a_{k,l}\,(t, v)\, x_{k,l} = x\ ,$$

it is necessary and sufficient that

(a) $$\lim_{t\to t_0, v\to v_0} a_{k,l}\,(t, v) = 0, \textit{ for each } k \textit{ and } l;$$

(b) $$\sum_{k=1,l=1}^{\infty,\infty} |a_{k,l}\,(t, v)| \textit{ converge for each } t \textit{ and } v;$$

(c) $$\lim_{t\to t_0, v\to v_0} \sum_{k=1,l=1}^{\infty,\infty} a_{k,l}\,(t, v) = 1;$$

(d) $$\lim_{t\to t_0, v\to v_0} \sum_{k=1}^{\infty} |a_{k,l}\,(t, v)| = 0, \textit{ for each } l;$$

(e) $$\lim_{t\to t_0, v\to v_0} \sum_{l=1}^{\infty} |a_{k,l}\,(t, v)| = 0, \textit{ for each } k;$$

(f) $$\sum_{k=1,l=1}^{\infty,\infty} |a_{k,l}\,(t, v)| < A \textit{ for every pair of values of } t \textit{ and } v.$$

THEOREM IV. *Let $a_{k,l}(t, v)$ be defined for $k=1, 2, 3, \cdots, l=1, 2, 3, \cdots$, t in T, v in V, where T and V are two point sets in the real or complex plane, having t_0 and v_0 (finite or infinite) respectively as limit points. Then in order that whenever the bounded sequence $(x_{m,n})$ possesses a finite limit x, it shall be true that for each pair of values t and v*

$$\sum_{k=1,l=1}^{\infty,\infty} a_{k,l}(t, v)x_{k,l} \text{ exists}$$

and that

$$\lim_{t\to t_0, v\to v_0} \sum_{k=1,l=1}^{\infty,\infty} a_{k,l}(t, v)\, x_{k,l} = \lambda x$$

(λ being a fixed constant), it is necessary and sufficient that

(a) $$\lim_{t\to t_0, v\to v_0} a_{k,l}(t, v) = 0 \text{ for each } k \text{ and } l;$$

(b) $$\sum_{k=1,l=1}^{\infty,\infty} |a_{k,l}(t, v)| \text{ converge for each } t \text{ and } v;$$

(c) $$\lim_{t\to t_0, v\to v_0} \sum_{k=1,l=1}^{\infty,\infty} a_{k,l}(t, v) = \lambda;$$

(d) $$\lim_{t\to t_0, v\to v_0} \sum_{k=1}^{\infty} |a_{k,l}(t, v)| = 0 \text{ for each } l;$$

(e) $$\lim_{t\to t_0, v\to v_0} \sum_{l=1}^{\infty} |a_{k,l}(t, v)| = 0, \text{ for each } k;$$

(f) $$\sum_{k=1,l=1}^{\infty,\infty} |a_{k,l}(t, v)| < A \text{ for all values of } t \text{ and } v.$$

Proof. By forming a new transformation

$$b_{k,l}(t, v) = \frac{a_{k,l}(t, v)}{\lambda},$$

when $\lambda \neq 0$, this problem reduces to that of the preceding theorem.

For $\lambda = 0$ the proof is reduced to that of the preceding theorem by the transformation defined as follows:

Let $\epsilon_1, \epsilon_2, \epsilon_3, \cdots$ be a monotonically decreasing sequence of real numbers and such that $\epsilon_n \to 0$.

(a) If t_0 and v_0 are finite limit points, then for

$$\epsilon_n \leqq |t-t_0| + |v-v_0| < \epsilon_{n-1}, \epsilon_0 = +\infty,$$

define

$$b_{n,n}(t, v) = a_{n,n}(t, v) + 1,$$

$$b_{k,l}(t, v) = a_{k,l}(t, v) \qquad \left\{\begin{matrix} k \neq n,\ l \neq n \\ k = n,\ l \neq n \\ k \neq n,\ l = n \end{matrix}\right\}.$$

(*b*) If t_0 is finite and v_0 infinite, then for

$$\epsilon_n \leqq |t - t_0| + \frac{1}{|v|} < \epsilon_{n-1}\ ,\ \epsilon_0 = +\infty,$$

define

$$b_{n,n}(t, v) = a_{n,n}(t, v) + 1,$$

$$b_{k,l}(t, v) = a_{k,l}(t, v) \qquad \left\{\begin{matrix} k \neq n,\ l \neq n \\ k = n,\ l \neq n \\ k \neq n,\ l = n \end{matrix}\right\}.$$

(*c*) If t_0 is infinite and v_0 finite, then for

$$\epsilon_n \leqq \frac{1}{|t|} + |v - v_0| < \epsilon_n - 1,\ \ \epsilon_0 = +\infty,$$

define

$$b_{n,n}(t, v) = a_{n,n}(t, v) + 1,$$

$$b_{k,l}(t, v) = a_{k,l}(t, v) \qquad \left\{\begin{matrix} k \neq m,\ l \neq n \\ k = n,\ l \neq n \\ k \neq n,\ l = n \end{matrix}\right\}.$$

(*d*) If t_0 and v_0 are both infinite, then for

$$\epsilon_n \leqq \frac{1}{|t|} + \frac{1}{|v|} < \epsilon_{n-1},\ \epsilon_0 = +\infty,$$

define

$$b_{n,n}(t, v) = a_{n,n}(t, v) + 1,$$

$$b_{k,l}(t, v) = a_{k,l}(t, v) \qquad \left\{\begin{matrix} k \neq n,\ l \neq n \\ k = n,\ l \neq n \\ k \neq n,\ l = n \end{matrix}\right\}.$$

As defined, the transformation $b_{k,l}(t, v)$ is regular whenever the transformation $a_{k,l}(t, v)$ satisfies the conditions of the theorem. If we write

$$y(t, v) = \sum_{k=1,l=1}^{\infty,\infty} b_{k,l}(t, v)x_{k,l} = \sum_{k=1,l=1}^{\infty,\infty} a_{k,l}(t, v)x_{k,l} + x_{n,n},$$

then by the preceding theorem

$$\lim_{t \to t_0, v \to v_0} y(t, v) = \lim_{t \to t_0, v \to v_0} \sum_{k=1,l=1}^{\infty,\infty} b_{k,l}(t, v)x_{k,l} = x.$$

Thus

$$\lim_{t\to t_0, v\to v_0}\left[\sum_{k=1,l=1}^{\infty,\infty} a_{k,l}(t,v)x_{k,l}+x_{n,n}\right]=x\ .$$

Hence

$$\lim_{t\to t_0, v\to v_0}\sum_{k=1,l=1}^{\infty,\infty} a_{k,l}(t,v)\ x_{k,l}+x=x,$$

which reduces to the expression

$$\lim_{t\to t_0, v\to v_0}\sum_{k=1,l=1}^{\infty,\infty} a_{k,l}(t,v)\ x_{k,l}=0.$$

Thus we have proved the theorem.

In the preceding work we assumed that the given sequence $x_{m,n}$ was bounded and convergent and then found the necessary and sufficient condition that the transformation must satisfy in order that the new sequence $y_{m,n}$ must converge to the same value. Different conditions may be placed upon the given sequence $x_{m,n}$ and then we ask what conditions must the transformation satisfy in order that the new sequence $y_{m,n}$ shall be bounded and convergent. The results of these investigations are stated without proof in the following theorems.

THEOREM V. *In order that a regular transformation of type T be totally regular it is necessary and sufficient that there exist intergers k_0 and l_0 such that $a_{m,n,k,l}>0$ when $k>k_0$, $l>l_0$, for all values of m and n.*

THEOREM VI. *A necessary and sufficient condition that a regular transformation of type T of real elements transform a bounded sequence $(x_{m,n})$, whose superior [inferior] limit is L [l], into a bounded sequence $(y_{m,n})$ whose superior [inferior] limit L' [l'] satisfies the relation $L'\leqq L$ $[l'\geqq l]$ is that the transformation be totally regular.*

THEOREM VII. *A necessary and sufficient condition that a transformation of type T transform a bounded convergent sequence $(x_{m,n})$ into a bounded convergent sequence $(y_{m,n})$ is that the following conditions hold:*

(a) $\lim_{m,n\to\infty} a_{m,n,k,l}$ *exists for each k and l;*

denote the value of the limit by $c_{k,l}$;

(b) $\lim_{m,n\to\infty}\sum_{k=1,l=1}^{m,n} a_{m,n,k,l}$ *exists;*

denote the value of this limit by a;

(c) $\sum_{k=1,l=1}^{m,n}|a_{m,n,k,l}|<A$ *for all m and n;*

(*d*) $$\lim_{m,n\to\infty} \sum_{k=1}^{m} |a_{m,n,k,l}-c_{k,l}|=0, \text{ for each } l;$$

(*e*) $$\lim_{m,n\to\infty} \sum_{l=1}^{n} |a_{m,n,k,l}-c_{k,l}|=0, \text{ for each } k.$$

When these conditions are satisfied, we have

$$\lim_{m,n\to\infty} y_{m,n}=ax+\sum_{k=1,l=1}^{\infty,\infty} c_{k,l}(x_{k,l}-x),$$

where $\lim_{m,n\to\infty} x_{m,n}=x$, *the series* $\sum_{k=1,l=1}^{\infty,\infty} c_{k,l}(x_{k,l}-x)$ *being always convergent.*

THEOREM VIII. *A necessary and sufficient condition that the sequence* $(y_{m,n})$ *defined by the relation*

$$y_{m,n}=\sum_{k=1,l=1}^{m,n} a_{m,n,k,l}\, x_{k,l}$$

shall converge whenever the bounded sequence $(x_{m,n})$ *converges and that the limit of* $(y_{m,n})$ *shall depend merely upon the limit of* $(x_{m,n})$ *and not upon the elements of* $(x_{m,n})$, *is that the following conditions hold:*

(*a*) $$\lim_{m,n\to\infty} a_{m,n,k,l}=0, \text{ for each } k \text{ and } l;$$

(*b*) $$\lim_{m,n\to\infty} \sum_{k=1,l=1}^{m,n} a_{m,n,k,l} \text{ exists};$$

(*c*) $$\sum_{k=1,l=1}^{m,n} |a_{m,n,k,l}|<A \text{ for all } m \text{ and } n \ (A \text{ being a fixed constant});$$

(*d*) $$\lim_{m,n\to\infty} \sum_{k=1}^{m} |a_{m,n,k,l}|=0, \text{ for each } l;$$

(*e*) $$\lim_{m,n\to\infty} \sum_{l=1}^{n} |a_{m,n,k,l}|=0, \text{ for each } k.$$

When these conditions are satisfied, we have

$$\lim_{m,n\to\infty} y_{m,n}=ax.$$

THEOREM IX. *A necessary and sufficient condition that a transformation of the type S transform a bounded convergent sequence* $(x_{m,n})$ *into a bounded convergent sequence* $(y_{m,n})$ *is that the following conditions hold:*

(*a*) $$\lim_{m,n\to\infty} a_{m,n,k,l} \text{ exists for each } k \text{ and } l;$$

denote the value of the limit by $c_{k,l}$;

(*b*) $$\lim_{m,n\to\infty} \sum_{k=1,l=1}^{\infty,\infty} a_{m,n,k,l} \text{ exists};$$

denote the value of this limit by a;

(*c*) $$\sum_{k=1,l=1}^{\infty,\infty} |a_{m,n,k,l}| < A \text{ for all } m \text{ and } n;$$

(*d*) $$\lim_{m,n\to\infty} \sum_{k=1}^{\infty} |a_{m,n,k,l} - c_{k,l}| = 0 \text{ for each } l;$$

(*e*) $$\lim_{m,n\to\infty} \sum_{l=1}^{\infty} |a_{m,n,k,l} - c_{k,l}| = 0 \text{ for each } k.$$

When these conditions are satisfied, we have

$$\lim_{m,n\to\infty} y_{m,n} = ax + \sum_{k=1,l=1}^{\infty,\infty} c_{k,l}(x_{k,l} - x),$$

where $x = \lim_{m,n\to\infty} x_{m,n}$, *the series* $\sum_{k=1,l=1}^{\infty,\infty} c_{k,l}(x_{k,l} - x)$ *being always absolutely convergent.*

THEOREM X. *A necessary and sufficient condition that the sequence* $(y_{m,n})$ *defined by the relation*

$$y_{m,n} = \sum_{k=1,l=1}^{\infty,\infty} a_{m,n,k,l}\, x_{k,l}$$

shall converge whenever the bounded sequence $(x_{m,n})$ *converges, and that the limit of* $(y_{m,n})$ *shall depend merely upon the limit of* $(x_{m,n})$ *and not upon the elements of* $(x_{m,n})$, *is that*

(*a*) $$\lim_{m,n\to\infty} a_{m,n,k,l} = 0 \text{ for each } k \text{ and } l,$$

(*b*) $$\sum_{k=1,l=1}^{\infty,\infty} |a_{m,n,k,l}| \text{ converge for each } m \text{ and } n,$$

(*c*) $$\lim_{m,n\to\infty} \sum_{k=1,l=1}^{\infty,\infty} a_{m,n,k,l} \text{ exist},$$

(*d*) $$\lim_{m,n\to\infty} \sum_{k=1}^{\infty} |a_{m,n,k,l}| = 0, \; l \text{ any fixed integer},$$

(*e*) $$\lim_{m,n\to\infty} \sum_{l=1}^{\infty} |a_{m,n,k,l}| = 0, \; k \text{ any fixed integer}.$$

When these conditions are satisfied, we have

$$\lim_{m,n\to\infty} y_{m,n} = ax .$$

THEOREM XI. *A necessary and sufficient condition that the transformation* $(a_{m,n,k,l})$ *of type T transform a bounded sequence* $(x_{m,n})$ *into a bounded convergent sequence* $(y_{m,n})$ *is that there exist numbers* $a_{k,l}$ *such that*

(i) $$\lim_{m,n\to\infty} \sum_{k=1,l=1}^{m,n} |a_{m,n,k,l} - a_{k,l}| = 0,$$

(ii) $$\sum_{k=1,l=1}^{\infty,\infty} |a_{k,l}| \text{ converges.}$$

THEOREM XII. *A necessary and sufficient condition that the transformation* $(a_{m,n,k,l})$ *of type S transform a bounded sequence* $(x_{m,n})$ *into a bounded convergent sequence* $(y_{m,n})$ *is that*

(i) $$\sum_{k=1,l=1}^{\infty,\infty} |a_{m,n,k,l}|$$

converge for each m and n and that there exist numbers $a_{k,l}$ *such that*

(ii) $$\lim_{m,n\to\infty} \sum_{k=1,l=1}^{\infty,\infty} |a_{m,n,k,l} - a_{k,l}| = 0,$$

(iii) $$\sum_{k=1,l=1}^{\infty,\infty} a_{k,l} \text{ converges.}$$

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ON CERTAIN FAMILIES OF ORBITS WITH ARBITRARY MASSES IN THE PROBLEM OF THREE BODIES*

BY

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This paper is devoted to the study of certain families of orbits in the problem of three bodies, which lie in the neighborhood of known orbits of simple types, but in which the masses are arbitrary and fixed. The classical method of variation of a parameter which enters into the equations of motion cannot, therefore, be employed; instead, extensive use is made of methods in which the notion of *invariant relation*† plays a prominent rôle.

In the first two paragraphs are developed certain implications regarding the equations of variation, which result from the existence of a set of invariant relations satisfied by the generating solution of a differential system, following which certain properties of isosceles triangle solutions with axis of symmetry in the problem of three bodies are established. The latter part of the paper is given up to the study of plane orbits which lie in the neighborhood of the straight line solutions, or in the neighborhood of the equilateral triangle solutions, respectively; in this study a simple reduction of the equations of motion in the plane is obtained, which is found especially convenient.

Many of the results obtained concerning these last solutions have been given by D. Buchanan,‡ but the method employed here has made it possible to go farther in certain respects. Certain related questions of stability will be taken up in a later paper.

1. Invariant relations

Suppose given a system of differential equations

$$\frac{dx_i}{dt} = X_i(x_1, x_2, \cdots, x_n) \qquad (i=1, \cdots, n) \tag{1}$$

* Presented to the Society, September 7, 1923; received by editors in January, 1924.

† See Poincaré, *Les Méthodes Nouvelles de la Mécanique Celeste*, vol. 1, p. 45.

‡ Rendiconti del Circolo Matematico di Palermo, vol. 25 (1921), pp. 1-25; these, Transactions, vol. 23 (1922), pp. 409-431.

in which X_i is analytic in its arguments, if $(x_1, \cdots, x_n)$ lies within a region D. Suppose the equations

$$\begin{aligned} &\varphi_1(x_1, \cdots, x_n)=0\ , \\ &\cdots\cdots\cdots \\ &\varphi_r(x_1, \cdots, x_n)=0 \end{aligned} \tag{2}$$

to form a set of invariant relations for solutions of (1). If a particular solution of (1) is defined by the equations

$$x_1=w_1(t)\ , \cdots, x_n=w_n(t)$$

in an interval $\delta: (0\leqq t\leqq T)$, let R be the region composed of all points $(x_1, \cdots, x_n)$ of D which satisfy the inequalities

$$|x_i-w_i(t)|<a \qquad (i=1, \cdots, n)\ , \tag{α}$$

for some t on δ. It will be assumed that $\varphi_1, \cdots, \varphi_r$ are analytic within R, and that for every value of t on δ some r-rowed determinant of the matrix

$$\left\| \begin{matrix} \dfrac{\partial\varphi_1}{\partial x_1} & \dfrac{\partial\varphi_1}{\partial x_2} & \cdots & \dfrac{\partial\varphi_1}{\partial x_n} \\ \dfrac{\partial\varphi_2}{\partial x_1} & \dfrac{\partial\varphi_2}{\partial x_2} & \cdots & \dfrac{\partial\varphi_2}{\partial x_n} \\ \cdots\cdots & \cdots\cdots & \cdots & \cdots\cdots \\ \dfrac{\partial\varphi_r}{\partial x_1} & \dfrac{\partial\varphi_r}{\partial x_2} & \cdots & \dfrac{\partial\varphi_r}{\partial x_n} \end{matrix} \right\|$$

is different from zero in the region defined by inequalities (α).

Under these hypotheses every value t^0 of t on δ lies in an interval $\delta': |t-t^0|<\eta$ such that the equations

$$\begin{aligned} &y_1=\varphi_1(x_1, \cdots, x_n)\ , \\ &\cdots\cdots\cdots \\ &y_r=\varphi_r(x_1, \cdots, x_n) \end{aligned}$$

can be solved for r of the variables $x_1, \cdots, x_n$; if these are $x_1, \cdots, x_r$,

$$x_i=\psi_i(y_1, \cdots, y_r, x_{r+1}, \cdots, x_n) \qquad (i=1, \cdots, r)$$

in a region

$$|y_i|<a', \quad |x_{r+k}-w_{r+k}(t)|<a'\ , \tag{β}$$

and inequalities (β) have (α) as a consequence if a' is suitably chosen,* t on δ'. Also, ψ_i is analytic in its arguments in the region defined by (β), if t is on δ'. If $x_{r+1}=y_{r+1}, \cdots, x_n=y_n$,

$$(1') \qquad \begin{aligned} \frac{dy_i}{dt} &= \sum_{k=1}^{n} X_k \frac{\partial \varphi_i}{\partial x_k} = Y_i(y_1, \cdots, y_n) && (i=1, \cdots, r), \\ \frac{dy_l}{dt} &= X_l(x_1, \cdots, x_n) = Y_l(y_1, \cdots, y_n) && (l=r+1, \cdots, n). \end{aligned}$$

The functions Y_i are analytic in their arguments, and may be expanded as power series in $y_1, \cdots, y_r$:

$$(\beta') \qquad \begin{aligned} &Y_i = \sum A^{(i)}_{a_1 \cdots a_r} y_1^{a_1} \cdots y_r^{a_r}, \\ &|y_i| < a'', \qquad |x_k - w_k(t)| < a'' \quad (i=1, \cdots, r; \quad k=r+1, \cdots, n). \end{aligned}$$

Since the equations (2) form a set of invariant relations, $Y_1, \cdots, Y_r$ must vanish when $y_1=y_2=\cdots=y_r=0$. Hence

$$(2') \qquad \begin{aligned} &A^{(i)}_{0 \cdots 0} = 0, \\ &\frac{dy_i}{dt} = a_{i1}y_1 + a_{i2}y_2 + \cdots + a_{ir}y_r + \psi_2^{(i)} && (i=1, \cdots, r) \end{aligned}$$

where $\psi_2^{(i)}$ is of at least the second degree with respect to $y_1, \cdots, y_r$. In terms of the original variables

$$(2'') \qquad \frac{d\varphi_i}{dt} = \sum_{k=1}^{r} A_{ik}\varphi_k + \psi_2^{(i)}(\varphi_1, \cdots, \varphi_r, x_{r+1}, \cdots, x_n),$$

---

* For, from the classical theorems on implicit functions it follows that corresponding to each value t^0 of t on δ, there exists a region

$$(\delta_1) \qquad |y_i| < b, \quad |x_k - w_k(t^0)| < b \qquad (i=1, \cdots, r; k=r+1, \cdots, n)$$

for some r variables $x_1, \cdots, x_r$, such that the functions

$$x_i = \Psi_i(y_1, \cdots, y_r, x_{r+1}, \cdots, x_n) \qquad (i=1, \cdots, r)$$

are defined within (δ_1) and inequalities (α) are a consequence of (δ_1). Suppose

$$a'=b/2, \quad |X_i| < M \qquad (i=r+1, \cdots, n).$$

Then if $|t-t_0|<b/2M$, from (1),

$$|w_k(t)-w_k(t^0)|<a' \qquad (k=r+1, \cdots, n).$$

Hence if

$$(\beta) \qquad |y_i| < a', |x_k - w_k(t)| < a', \quad |t-t_0| < a'/M \qquad (i=1, \cdots, r; \ k=r+1, \cdots, n),$$

it follows that $|x_k-w_k(t^0)|<b$, and inequalities (δ_1) are a consequence of (β).

and the series on the right must converge if $(x_1, \cdots, x_n)$ lies within a region

$$(\alpha') \qquad |x_i - w_i(t)| < \bar{a}'', \quad |t - t^0| < \eta \quad (i = 1, \cdots, n),$$

which corresponds to that defined by inequalities (β').

Since every t on δ lies in an interval δ' for which equations of the form $(2'')$ hold, it follows from the Heine-Borel theorem that δ can be covered by a finite number of intervals of this form, and these can be chosen as overlapping intervals δ'; if $\bar{a}$ is the smallest of the quantities a'', inequalities (α') are a consequence of inequalities

$$(\bar{\alpha}): \qquad |x_i - w_i(t)| < \bar{a}, \quad 0 \leqq t \leqq T.$$

Since the coefficients of the series on the right of $(2'')$, defined for a region δ', must coincide with the corresponding ones defined for δ'' in the region common to δ' and δ'', each must be the analytic prolongation of the other. It follows that *equations* $(2'')$ *are valid under the hypotheses concerning* X_i, φ_k, *within the region defined by inequalities* $(\bar{\alpha})$.

Now if

$$X(f) = \sum_{i=1}^{n} X_i \frac{\partial f}{\partial x_i},$$

then if $f(x_1, \cdots, x_n)$ is defined as a function of t by means of equations (1),

$$f(t) = f(x_1^0, \cdots, x_n^0) + tX(f) + \frac{t^2}{2!} X[X(f)] + \cdots.$$

If $\varphi_i(x_1, \cdots, x_n)$ is expanded in this manner, and if $\varphi_i(x_1^0, \cdots, x_n^0) = 0$ $(i = 1, \cdots, r)$, from $(2'')$ $X(\varphi_i) = 0$, $X[X(\varphi_i)] = 0, \cdots$ and hence $\varphi_i = 0$, for all values of t on δ. This result, announced by Poincaré (loc. cit.), seems of sufficient importance to warrant the detailed demonstration given above.

If the change of variables

$$y_1 = \varphi_1, \cdots, y_r = \varphi_r, \quad y_{r+1} = y_{r+1}^0 + z_{r+1}, \cdots, y_n = y_n^0 + z_n$$

is made, where $y_{r+1}^0, \cdots, y_n^0$ are a solution of the system

$$\frac{dy_i^0}{dt} = Y_i(0, \cdots, 0, y_{r+1}^0, \cdots, y_n^0) \qquad (i = r+1, \cdots, n),$$

we obtain

$$(3) \qquad \frac{dz_k}{dt} = \sum_{s=1}^{r} \frac{\partial Y_k}{\partial y_s} y_s + \sum_{s=r+1}^{n} \frac{\partial Y_k}{\partial y_s} z_s + \psi_2^{(k)}(y_1, \cdots, y_r, z_{r+1}, \cdots, z_n).$$

The equations of variation of the system (1′) are therefore in the form

$$(a)\quad \frac{d\bar{y}_i}{dt} = a_{i1}\bar{y}_1 + a_{i2}\bar{y}_2 + \cdots + a_{ir}\bar{y}_r \qquad (i=1, \cdots, r),$$

(4)

$$(b)\quad \frac{d\bar{z}_k}{dt} = b_{k1}\bar{y}_1 + \cdots + b_{kr}\bar{y}_r + b_{k,r+1}\bar{z}_{r+1} + \cdots + b_{kn}\bar{z}_n \qquad (k=r+1, \cdots, n).$$

If the system (a) is integrated first, (b) can be integrated as a non-homogeneous system in $\bar{z}_{r+1}, \cdots, \bar{z}_n$.

The system obtained from (1′) by giving $y_1, \cdots, y_r$ the value zero will be called the reduced system with respect to the relations (2). This reduced system may admit another set of invariant relations, by means of which a second reduced system is obtained, etc. In this manner any given set of invariant relations can be grouped into sub-sets,

$$S_1 : \quad (\varphi_1', \cdots, \varphi_{r_1}'),$$

$$S_2 : \quad (S_1, \varphi_1'', \cdots, \varphi_{r_2}''),$$

$$\cdots\cdots\cdots$$

$$S_m : \quad (S_{m-1}, \overset{(m)}{\varphi_1}, \cdots, \overset{(m)}{\varphi_{r_m}}),$$

such that when the functions of the set S_{k-1} are set equal to zero, the relations

$$\overset{(k)}{\varphi_1} = 0, \cdots, \overset{(k)}{\varphi_{r_k}} = 0$$

form a set of invariant relations. This division into sub-sets can in general be made in a variety of ways.

It is seen immediately that a corresponding grouping of the equations of (a) can be made; these are necessarily in the form (if $s_k = r_1 + r_2 + \cdots + r_k$)

$$(S_1')\quad \frac{d\bar{y}_i'}{dt} = \sum_{k=1}^{r_1} a_{ik}\bar{y}_k' \qquad (i=1, \cdots, r_1),$$

$$(S_2')\quad \frac{d\bar{y}_i''}{dt} = \sum_{k=1}^{r_1} a_{s_1+i,k}\bar{y}_k' + \sum_{k=1}^{r_2} a_{s_1+i,s_1+k}\bar{y}_k'' \qquad (i=1, \cdots, r_2),$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$(S_m')\quad \frac{d\bar{y}_i^{(m)}}{dt} = \sum_{k=1}^{r_1} a_{s_{m-1}+i,k}\bar{y}_k' + \sum_{k=1}^{r_2} a_{s_{m-1}+i,s_1+k}\bar{y}_k'' + \cdots + \sum_{k=1}^{r_m} a_{s_{m-1}+i,s_{m-1}+k}\bar{y}_k^{(m)} \qquad (i=1, 2, \cdots, r_m).$$

The integration of equations (a) is therefore reduced to the successive integrations of the sets (S_1'), (S_2'), $\cdots$, (S_m'), the set (S_k') forming a non-homogeneous system in $\bar{y}_1^{(k)}, \cdots, \bar{y}_{r_k}^{(k)}$, if the sets $(S_1'), \cdots, (S_{k-1}')$ have been previously integrated.

2. A special form of invariant relations

Suppose equations (1, §1) in the canonical form

$$(1') \qquad \begin{aligned} \frac{dx_i}{dt} &= \frac{\partial F}{\partial y_i}, \quad \frac{dy_i}{dt} = -\frac{\partial F}{\partial x_i} && (i=1, \cdots, m), \\ \frac{du_i}{dt} &= \frac{\partial F}{\partial v_i}, \quad \frac{dv_i}{dt} = -\frac{\partial F}{\partial u_i} && (i=1, \cdots, r), \end{aligned}$$

and suppose the equations

$$(1) \qquad u_1 = u_2 = \cdots = u_r = v_1 = v_2 = \cdots = v_r = 0$$

a set of invariant relations. Then the functions $\partial F/\partial u_i$, $\partial F/\partial v_k$ are of at least the first degree in u_i, v_i, and the equations of variation for $(1')$ are

$$(2) \qquad \begin{aligned} \frac{d\xi_i}{dt} &= \sum_{k=1}^{m} \frac{\partial^2 F}{\partial y_i \partial x_k}\xi_k + \sum_{k=1}^{m} \frac{\partial^2 F}{\partial y_i \partial y_k}\eta_k, \\ \frac{d\eta_i}{dt} &= -\sum_{k=1}^{m} \frac{\partial^2 F}{\partial x_i \partial x_k}\xi_k - \sum_{k=1}^{m} \frac{\partial^2 F}{\partial x_i \partial y_k}\eta_k \quad (i=1, \cdots, m); \end{aligned}$$

$$(3) \qquad \begin{aligned} \frac{d\bar{u}_i}{dt} &= \sum_{k=1}^{r} \frac{\partial^2 F}{\partial v_i \partial u_k}\bar{u}_k + \sum_{k=1}^{r} \frac{\partial^2 F}{\partial v_i \partial v_k}\bar{v}_k, \\ \frac{d\bar{v}_i}{dt} &= -\sum_{k=1}^{r} \frac{\partial^2 F}{\partial u_i \partial u_k}\bar{u}_k - \sum_{k=1}^{r} \frac{\partial^2 F}{\partial u_i \partial v_k}\bar{v}_k \quad (i=1, \cdots, r). \end{aligned}$$

3. Periodic orbits

In this paragraph will be outlined a variation of the method of Poincaré, for determining the periodic solutions of a differential system, which does not contain a parameter explicitly.

Suppose

$$\bar{x}_i = \varphi_i^0(t), \quad \varphi_i^0(0) = x_i^0 \qquad (i=1, \cdots, n)$$

a periodic solution of the system

$$(1) \qquad \frac{dx_i}{dt} = X_i(x_1, x_2, \cdots, x_n) \qquad (i=1, \cdots, n),$$

and consider a solution

$$x_i=\varphi_i(t), \quad \varphi_i(0)=x_i^0+\beta_i \qquad (i=1, \cdots, n);$$

the functions φ_i are analytic in $\beta_1, \cdots, \beta_n$ for $|\beta_i|<\epsilon$, under the hypotheses of the first paragraph, if ϵ is sufficiently small, and can be expanded as power series in these variables.*

The expansion of x_i is in the form

$$(2) \qquad x_i=x_i^0(t)+\sum_{k=1}^{n}\xi_i^{(k)}(t)\beta_k+\Phi_2^{(i)}(t;\beta_1, \cdots, \beta_n) \qquad (i=1, \cdots, n)$$

where $\Phi_2^{(i)}$ is of at least the second degree with respect to $\beta_1, \cdots, \beta_n$, the coefficients being continuous functions of t, and the series converge if $0\leqq t\leqq \bar{T}$, $|\beta_i|<\epsilon$ $(i=1, 2, \cdots, n)$.

The functions $\xi_i^{(k)}(t)$ are solutions of the equations of variation for (1),

$$(3) \qquad \frac{d\xi_i^{(k)}}{dt}=\sum_{s=1}^{n}\frac{\partial X_i}{\partial x_s}\xi_s^{(k)} \qquad (i=1, \cdots, n;\ k=1, \cdots, n),$$

and are such that for $t=0$,

$$(4) \qquad \xi_i^{(k)}(0)=\delta_i^{(k)} \qquad (\delta_i^{(i)}=1, \quad \delta_i^{(k)}=0, \quad k\neq i).$$

If the generating periodic orbit has the period T, the problem of determining periodic orbits of period $T+\tau$ is reduced to that of solving the system of equations

$$(5) \qquad \Delta x_i=x_i(T+\tau)-x_i(0)=0 \qquad (i=1, \cdots, n),$$

for $\beta_1, \cdots, \beta_n, \tau$. From (2),

$$\Delta x_i=x_i'(0)\tau+\sum_{k=1}^{n}\Delta\xi_i^{(k)}\beta_k+\psi_2^{(i)}(\tau, \beta_1, \cdots, \beta_n),$$

where

$$\Delta\xi_i^{(k)}=\xi_i^{(k)}(T)-\xi_i^{(k)}(0) \qquad (i=1, \cdots, n;\ \ k=1, \cdots, n),$$

and $\psi_2^{(i)}$ is of at least the second degree in $\tau, \beta_1, \cdots, \beta_n$.

Equations (5) can be solved for small values of $\tau, \beta_1, \cdots, \beta_n$ if not all n-rowed determinants of the matrix

$$A\equiv \left\| \begin{matrix} X_1^0, & \Delta\xi_1', & \Delta\xi_1'', & \cdots, & \Delta\xi_1^{(n)} \\ X_2^0, & \Delta\xi_2', & \Delta\xi_2'', & \cdots, & \Delta\xi_2^{(n)} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ X_n^0, & \Delta\xi_n', & \Delta\xi_n'', & \cdots, & \Delta\xi_n^{(n)} \end{matrix} \right\| \qquad (x_i'(0)=X_i^0=X_i(x_1^0, \cdots, x_n^0))$$

* *Les Méthodes Nouvelles*, vol. 1, Chap. 2.

are zero. If a certain number of uniform integrals of (1) exist, the rank of A will in general be less than n; but in this case* these integrals can be used in place of some of the equations (5).

4. Certain properties of isosceles triangle solutions with axis of symmetry, in the problem of three bodies

The equations of motion of the isosceles triangle solutions with axis of symmetry in the problem of three bodies can be given the form

$$
\begin{aligned}
&\frac{dr}{dt} = \frac{\partial F}{\partial R}, \qquad \frac{d\zeta}{dt} = \frac{\partial F}{\partial Z},\\
&\frac{dR}{dt} = -\frac{\partial F}{\partial r}, \qquad \frac{dZ}{dt} = -\frac{\partial F}{\partial \zeta},\\
&F = \frac{1}{2m'}\left(R^2 + \frac{\Theta^2}{r^2}\right) + \frac{Z^2}{2\bar{M}} - \frac{4m'm_2}{r_0} - \frac{4m'^2}{r},\\
&\bar{M} = \frac{4m'm_2}{4m' + m_2}, \quad r_0 = \sqrt{\zeta^2 + (r/2)^2}.
\end{aligned}
\tag{1}
$$

In these equations $2m'$ is the common mass of two of the bodies, m_2 the mass of the third, r the distance between the two equal masses, r_0 the distance from either to the third, and ζ is the distance from the third to the center of gravity of the first two.

These equations admit the integral $F = C$, and it was shown by J. Chazy† that if $C < 0$, $\Theta \neq 0$, r must satisfy certain inequalities

$$0 < \alpha \leqq r \leqq \beta . \tag{2}$$

It can also be shown that positive constants $\bar{A}$, $\bar{B}$ can be found such that if $\Theta \neq 0$, $C < -\bar{A}$, then $|\zeta| < \bar{B}$ and consequently

$$0 < \gamma < r_0 < \delta \tag{3}$$

for certain constants γ, δ and for all values of t.

For from the energy integral, if $C = -C'$,

$$
\begin{aligned}
&\frac{1}{2m'}\left(R^2 + \frac{\Theta^2}{r^2}\right) + \frac{Z^2}{2\bar{M}} - \frac{4m'm_2}{r_0} - \frac{4m'^2}{r} = -C',\\
&\frac{R^2}{2m'} + \frac{Z^2}{2\bar{M}} = \frac{4m'm_2}{r_0} + \frac{4m'^2}{r} - \frac{\Theta^2}{2m'r^2} - C' \geqq 0 .
\end{aligned}
$$

* *Les Méthodes Nouvelles*, vol. 1, Chap. 3.

† Bulletin Astronomique, vol. 1, No. 3, 1921.

Assume $\Theta \neq 0$; then

$$-\frac{1}{r^2}+\frac{8m'^3}{\Theta^2 r} \geqq \frac{2m'C'}{\Theta^2}-\frac{8m'^2m_2}{\Theta^2 r_0},$$

from which

$$-\left(\frac{1}{r}-\frac{4m'^3}{\Theta^2}\right)^2 \geqq \frac{2m'C'}{\Theta^2}-\left(\frac{4m'^3}{\Theta^2}\right)^2-\frac{8m'^2m_2}{\Theta^2 r_0},$$

$$\frac{4m'^2m_2}{r_0} \geqq \frac{m'C'\Theta^2-8m'^6}{\Theta^2};$$

or if $C'\Theta^2-8m'^5>0$,

$$|\zeta| \leqq r_0 \leqq \frac{4m'm_2\Theta^2}{C'\Theta^2-8m'^5}.$$

If the variable t is replaced by θ, from the relation

$$m'r^2\frac{d\theta}{dt}=\Theta, \qquad d\theta=\frac{\Theta}{m'r^2}\,dt,$$

equations (1) become

$$(1') \qquad \begin{aligned} \frac{dr}{d\theta} &= \frac{m'r^2}{\Theta}\frac{\partial F}{\partial R}, & \frac{d\zeta}{d\theta} &= \frac{m'r^2}{\Theta}\frac{\partial F}{\partial Z}, \\ \frac{dR}{d\theta} &= -\frac{m'r^2}{\Theta}\frac{\partial F}{\partial r}, & \frac{dZ}{d\theta} &= -\frac{m'r^2}{\Theta}\frac{\partial F}{\partial \zeta}, \end{aligned}$$

and these equations admit the multiplier $M=\Theta/m'r^2$, where M is positive and finite so long as the initial conditions are such that

$$C'\Theta^2-8m'^5>C_1^2>0;$$

while from the assumption $C'<0$ it follows that

$$r \geqq \frac{\Theta^2}{2m'(8m'm_2+4m'^2)},$$

an inequality easily obtained from the energy integral and the inequality $1/r \geqq 1/2r_0$.

It follows that for some region

$$(D) \qquad \begin{aligned} 0<a_1<|r|<b_1, \qquad & |R|<B_1, \\ |\zeta|<\bar{B}, \qquad & |Z|<\bar{C} \end{aligned}$$

for which

$$C_2^2>C'\Theta^2-8m'^5>C_1^2>0,$$

the variables r, R, ζ, Z lie in (D) for all values of t, and the equations (1′) remain valid; since these equations admit a positive finite multiplier the solutions possess *complete stability*.*

From (1) we obtain

$$\frac{d\zeta}{dt}=\frac{Z}{\bar{M}}, \qquad \frac{dZ}{dt}=-\frac{4m'm_2}{r^3}\zeta,$$

or if $a=\sqrt{4m'm_2/\bar{M}}$,

$$\frac{d^2\zeta}{dt^2}+\frac{a^2}{r_0^3}\zeta=0. \tag{4}$$

From (3),

$$\frac{a^2}{\delta^3}<\frac{a^2}{r_0^3}<\frac{a^2}{\gamma^3}. \tag{5}$$

Consider an equation

$$\frac{d^2\zeta}{dt^2}+\varphi(t,\zeta)\zeta=0,$$

in which

$$A^2<\varphi(t,\zeta)<B^2, \qquad |\zeta|<\epsilon,$$

for all values of t, A and B being certain positive constants, and φ continuous in its arguments. Let z_1, z_2 be solutions of the equations

$$\frac{d^2z_1}{dt^2}+B^2z_1=0,$$

$$\frac{d^2z_2}{dt_2}+A^2z_2=0.$$

Then

$$z_1\zeta''-\zeta z_1''=\zeta z_1[B^2-\varphi(t,\zeta)],$$

$$\zeta z_2''-z_2\zeta''=\zeta z_2[\varphi(t,\zeta)-A^2].$$

If z_1, z_2, ζ satisfy the initial conditions

$$z_1(0)=z_2(0)=\zeta(0)=0,$$

$$z_1'(0)>0, \quad z_2'(0)>0, \quad \zeta'(0)>0,$$

then

$$[z_1\zeta'-\zeta z_1']_0^{t_1}=\int_0^{t_1}[B^2-\varphi(t,\zeta)]z_1\zeta\,dt, \tag{6}$$

* Poincaré, *Les Méthodes Nouvelles*, vol. 3, p. 141. For the method of Poincaré can be applied to prove that if $P\{r_0, \zeta_0, R_0, Z_0\}$ is any point of (D), δ any region surrounding P, Q an arbitrary point of δ, θ_0 a given value of θ, the probability that (r, ζ, R, Z) will lie in δ only a finite number of times with $\theta=\theta_0+2k\pi$ is arbitrarily small (k an integer) if $(r, \zeta, R, Z)=Q$, $\theta=\theta_0$.

or if $z_1(t_1)=0$, t_1 being the first zero of $z_1(t)$ after $t=0$,

$$-\zeta(t_1)z_1'(t_1)>0\,,$$

and since $z_1'(t_1)<0$, $\zeta(t_1)>0$. In the same manner we obtain

$$\zeta(\bar{t})z_2'(\bar{t})-z_2(\bar{t})\zeta'(\bar{t}) = \int_0^{\bar{t}} [\varphi(t,\zeta)-A^2]z_2\zeta\,dt\,, \tag{7}$$

and if $\zeta(\bar{t})=0$, $\bar{t}$ being the first zero of $\zeta(t)$ greater than $t=0$,

$$-\zeta'(\bar{t})z_2(\bar{t})>0\,,$$

from which $z_2(\bar{t})>0$. Consequently if $\zeta(0)=0$, the first following zero of $\zeta(t)$ must lie between the first zero of $z_1(t)$ and that of $z_2(t)$. If this zero is $\bar{t}$,

$$\frac{\pi}{B}<\bar{t}<\frac{\pi}{A}\,;$$

applying this result to equation (4), with conditions (5),

$$\frac{\pi\gamma^{3/2}}{a}<\bar{t}<\frac{\pi\delta^{3/2}}{a}\,. \tag{8}$$

From (6) it follows also that if $0<t<\pi/B$,

$$\frac{z_1\zeta'-\zeta z_1'}{z_1^2}>0\,,$$

consequently if $0<t_1<t_2<\pi/B$,

$$\left[\frac{\zeta}{z_1}\right]_{t_1}^{t_2}=\int_{t_1}^{t_2}\frac{z_1\zeta'-\zeta z_1'}{z_1^2}\,dt>0\,. \tag{9}$$

Now suppose $z_1'(0)=\zeta'(0)$; for small values of t,

$$\zeta=\zeta_0't+t^3\psi(t)\,,$$

$$z_1=\frac{\zeta_0'}{B}\sin Bt\,.$$

From (9),

$$\frac{\zeta(t_2)}{z_1(t_2)}>\frac{\zeta_0't_1+t_1^3\psi(t_1)}{\frac{\zeta_0'}{B}\sin Bt_1}\,. \tag{10}$$

Taking the limit of the expression on the right of (10) as t_1 approaches zero,

$$\frac{\zeta(t_2)}{z_1(t_2)}>1\ ,$$

since the right member of (10) decreases as t_1 approaches zero.

An exactly analogous argument shows that if $\zeta(t)>0$, $0<t<\bar{t}$, and $z_2'(0)=\zeta'(0)$,

$$\frac{z_2(t)}{\zeta(t)}>1\ ,$$

or

$$z_1(t_2)<\zeta\ (t_2)\ , \qquad 0<t_2<\frac{\pi}{B}\ ,$$

$$\zeta\ (t)<z_2(t)\ , \qquad 0<t<\bar{t}\ .$$

Consequently* if $\bar{\zeta}$ is the maximum value of ζ on the interval $0<t<\bar{t}$,

$$\frac{\zeta_0'}{B}\leqq\bar{\zeta}\leqq\frac{\zeta_0'}{A}\ ,$$

or

$$\frac{1}{B}\leqq\frac{\bar{\zeta}}{\zeta_0'}\leqq\frac{1}{A}\ .$$

5. Periodic isosceles triangle solutions

Suppose the function F in equations (1,§4) expanded in powers of ζ; since

$$\frac{1}{r_0}=\frac{2}{r}\left[1+\left(\frac{2\zeta}{r}\right)^2\right]^{-1/2}$$

$$=\frac{2}{r}\left[1-\frac{1}{2}\left(\frac{2\zeta}{r}\right)^2+\frac{1\cdot 3}{2\cdot 4}\left(\frac{2\zeta}{r}\right)^4-\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\left(\frac{2\zeta}{r}\right)^6+\cdots\right]$$

we obtain

$$F=\frac{1}{2m'}\left(R^2+\frac{\Theta^2}{r^2}\right)-\frac{4m'^2+8m'm_2}{r}$$

$$+\frac{Z^2}{2\bar{M}}+\frac{16m'm_2}{r^3}\,\zeta^2 \tag{1}$$

$$-\frac{8m'm_2}{r}\left[\frac{1\cdot 3}{2\cdot 4}\left(\frac{2\zeta}{r}\right)^4-\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\left(\frac{2\zeta}{r}\right)^6+\cdots\right]$$

$$=F_0+F_2+F_4\ ,$$

* That $\zeta\ (t)$ must have at least one zero can be shown easily from the equation

$$\left[\zeta z_2'-z_2\zeta'\right]_{t_1}^{t_2}=\int_{t_1}^{t_2}[\varphi-A^2]\zeta z_2 dt,$$

by the method above, if $z_2(t_1)=\zeta(t_1)>0$, $z_2'(t_1)=\zeta'(t_1)$.

where F_0 is independent of ζ, Z, F_2 is of the second degree, and F_4 of at least the fourth degree with respect to these variables. The equations $\zeta=Z=0$ form evidently a set of invariant relations for equations (1, §4). If $\mu=8m_2+4m'$,

$$F_0=\frac{1}{2m'}\left(R^2+\frac{\Theta^2}{r^2}\right)-\frac{m'\mu}{r}, \tag{2}$$

and equations (1, §4), reduced by the relations $\zeta=Z=0$, can be integrated by the method of Jacobi. A complete integral of the partial differential equation

$$\frac{1}{2m'}\left[\left(\frac{\partial S}{\partial r}\right)^2+\frac{\Theta^2}{r^2}\right]-\frac{m'\mu}{r}=-\frac{m'^3\mu^2}{2L^2} \tag{3}$$

is the function

$$S=\int\sqrt{-\frac{\Theta^2}{r^2}+\frac{2m'^2\mu}{r}-\frac{m'^4\mu^2}{L^2}}\,dr,$$

where Θ, L are constants. If

$$R=\frac{\partial S}{\partial r}, \qquad l=\frac{\partial S}{\partial L},$$

equations (1, §4) become

$$\begin{aligned}
&\frac{dl}{dt}=\frac{\partial F}{\partial L}, \qquad \frac{d\zeta}{dt}=\frac{\partial F}{\partial Z},\\
&\frac{dL}{dt}=-\frac{\partial F}{\partial l}, \qquad \frac{dZ}{dt}=-\frac{\partial F}{\partial \zeta},\\
&F=-\frac{m'^3\mu^2}{2L^2}+F_2+F_4.
\end{aligned} \tag{4}$$

One might be tempted to employ the method of §3 to determine periodic solutions of (4). Suppose

$$\begin{aligned}
\Delta\zeta&=\zeta(T+\tau)-\zeta(0),\\
\Delta Z&=Z(T+\tau)-Z(0),\\
\psi&=l(T+\tau)-n_0T, \qquad n_0=\frac{m'^3\mu^2}{L^3},\\
\Delta L&=L(T+\tau)-L(0),
\end{aligned}$$

if the reduced system has a periodic solution of period $T=2\pi/n_0$. Since $F=C$ is an integral for (4), the equation $\Delta L=0$ can be omitted, and equations

$$\Delta\zeta=\Delta Z=\psi=0$$

can be solved for $\delta\zeta^0$, δZ^0, δL^0 in terms of τ if the determinant

$$J = \begin{vmatrix} T\dfrac{\partial n}{\partial L} & 0 & 0 \\ 0 & \Delta\bar{\zeta}_1 & \Delta\bar{\zeta}_2 \\ 0 & \Delta\bar{Z}_1 & \Delta\bar{Z}_2 \end{vmatrix}$$

is different from zero; $\bar{\zeta}_i$, $\bar{Z}_i$ are solutions of the equations of variation for the system (4),

$$\frac{d\bar{\zeta}}{dt} = \frac{\bar{Z}}{\bar{M}}, \qquad \frac{d\bar{Z}}{dt} = -\frac{32m'm_2}{r^3}\bar{\zeta}, \tag{5}$$

from which

$$\frac{d^2\bar{\zeta}}{dt^2} + \frac{32m'm_2}{\bar{M}r^3}\bar{\zeta} = 0. \tag{6}$$

The determinant J can vanish only if at least one of the characteristic exponents of (5) is of the form*

$$\alpha = \frac{2\kappa\pi i}{T} = \kappa n i,$$

where κ is an integer. A simple calculation shows that

$$\frac{\alpha}{n} = \pm i\sqrt{\frac{2(m_2+4m')}{2m_2+m'}}$$

if the eccentricity is zero for the reduced orbit. If $\alpha/n = \kappa i$,

$$\frac{m'}{m_2} = \frac{2(\kappa^2-1)}{8-\kappa^2}.$$

Now since m', m_2 are positive, the only possible value for κ is 2 and

$$\frac{m'}{m_2} = \frac{3}{2}. \tag{7}$$

The characteristic exponents for the system (5) are continuous functions of the eccentricity e in a region $|e| < h$, for some $h > 0$, hence if $J \neq 0$ for $e = 0$, $J \neq 0$ for an interval $|e| < h_1 \leqq h$. Consequently if the equation (7) is not satisfied, periodic solutions of period $T' = T + \tau$ must exist for a certain region $|\tau| < \epsilon$, $|e| < h_1$.

* *Les Méthodes Nouvelles*, vol. 1, p. 181.

But these are necessarily isosceles triangle solutions, for which $\zeta = Z = 0$. In fact periodic isosceles triangle solutions in the plane exist, with $\zeta = Z = 0$, and for these solutions the eccentricity of the relative elliptic movement can have any value between zero and one. From the equations satisfied by a periodic solution it is seen that there can be only one family of periodic isosceles triangle solutions, hence no periodic solutions except the plane isosceles triangle solutions, in general. For any other family of periodic solutions of the type considered, the ratio α/in is necessarily commensurable, where $\pm\alpha$ are the characteristic exponents, $2\pi/n$ the period of the generating plane isosceles triangle solution.

The plane isosceles triangle solutions (for which $\zeta = Z = 0$) can be still further particularized by substituting $\Theta = m'\sqrt{\mu r_1}$, for which value the equations $r = r_1$, $R = 0$ form a set of invariant relations. If we make the change of variables $r = r_1 + \eta$, the equations of motion, for small values of η and ζ, can be written

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \eta'}\right) = \frac{\partial U}{\partial \eta}, \quad \frac{d}{dt}\left(\frac{\partial T}{\partial \zeta'}\right) = \frac{\partial U}{\partial \zeta},$$

if

$$T = \tfrac{1}{2}[m'\eta'^2 + \bar{m}\zeta'^2],$$

$$U = -\frac{m'\mu}{2r_1^3}\eta^2 - \frac{16m'm_2}{r_1^3}\zeta^2 + \phi_3(\eta, \zeta).$$

It follows immediately from the classical theorem of Lagrange that the generating periodic solution is stable (within the family of isosceles triangle solutions).

6. Certain properties of the characteristic exponents

Suppose the movement of a material system defined by the equations

$$(1)\qquad \begin{aligned} &\frac{dx_i}{dt} = \frac{\partial F}{\partial y_i}, \quad \frac{dy_i}{dt} = -\frac{\partial F}{\partial x_i} && (i = 1, 2, \cdots, m), \\ &\frac{du_k}{dt} = \frac{\partial F}{\partial v_k}, \quad \frac{dv_k}{dt} = -\frac{\partial F}{\partial u_k} && (k = 1, 2, \cdots, r), \end{aligned}$$

where F is periodic, of period T in t, and analytic in its remaining $(2m+2r)$ variables within a certain domain D. If a periodic solution of period T, lying within D, is defined by the equations

$$(c)\qquad \begin{aligned} &x_i = x_i^0(t), \quad y_i = y_i^0(t) && (i = 1, \cdots, m), \\ &u_k = u_k^0(t), \quad v_k = v_k^0(t) && (k = 1, \cdots, r), \end{aligned}$$

many of the properties of orbits within a certain neighborhood of (c) can be obtained from a study of the equations of variation corresponding to this orbit.

Suppose $u_k^0(t) \equiv v_k^0(t) \equiv 0$, $k=1, \cdots, r$, $\partial F/\partial t = 0$, for (c), and in addition suppose $u_k = v_k = 0$ a set of invariant relations. It has already been seen that under these conditions the equations of variation break up into two or more distinct systems (§§1, 2). The characteristic exponents of each system are also characteristic exponents of (c).

If the variables u_i are integrals of equations (1), then on account of the relations $\partial F/\partial v_k \equiv 0$, $k=1, \cdots, r$, the equations of variation have the form

$$(\alpha)\qquad \frac{d\xi_i}{dt} = \sum_{k=1}^{m} \frac{\partial^2 F}{\partial y_i \partial x_k}\xi_k + \sum_{k=1}^{m} \frac{\partial^2 F}{\partial y_i \partial y_k}\eta_k + \sum_{k=1}^{r} \frac{\partial^2 F}{\partial y_i \partial u_k}\mathfrak{u}_k ,$$

$$\frac{d\eta_i}{dt} = -\sum_{k=1}^{m} \frac{\partial^2 F}{\partial x_i \partial x_k}\xi_k - \sum_{k=1}^{m} \frac{\partial^2 F}{\partial x_i \partial y_k}\eta_k - \sum_{k=1}^{r} \frac{\partial^2 F}{\partial x_i \partial u_k}\mathfrak{u}_k \qquad (i=1, \cdots, m) ; \tag{2}$$

$$(\beta)\qquad \frac{d\mathfrak{u}_i}{dt} = 0 ,$$

$$(\gamma)\qquad \frac{d\bar{v}_i}{dt} = -\sum_{k=1}^{m} \frac{\partial^2 F}{\partial u_i \partial x_k}\xi_k - \sum_{k=1}^{m} \frac{\partial^2 F}{\partial u_i \partial y_k}\eta_k - \sum_{k=1}^{r} \frac{\partial^2 F}{\partial u_i \partial u_k}\mathfrak{u}_k \qquad (i=1, \cdots, r) .$$

These equations can be integrated in the order (β), (α), (γ), the systems (α), (γ) being considered as non-homogeneous systems.

Suppose given a non-homogeneous linear system with periodic coefficients, which are also assumed analytic:

$$\frac{dx_i}{dt} = \sum_{k=1}^{n} a_{ik}x_k + f_i \qquad (i=1, \cdots, n) , \tag{3}$$

for which

$$\sum_{i=1}^{n} a_{ii} = 0 .$$

Suppose an independent set of solutions of the homogeneous system corresponding to (3) represented by the set of columns of the matrix

$$\bar{\Delta} = \begin{Vmatrix} x_1' & x_1'' & \cdots & x_1^{(n)} \\ x_2' & x_2'' & \cdots & x_2^{(n)} \\ \cdots & \cdots & \cdots & \cdots \\ x_n' & x_n'' & \cdots & x_n^{(n)} \end{Vmatrix} .$$

Then $\Delta = |x_i^{(k)}|$ is constant and different from zero, as a consequence of the classical theorems on linear differential systems. If $\Delta_i^{(k)}$ is the cofactor of $x_i^{(k)}$ in Δ, an arbitrary solution of equations (3) can be represented in the form

$$(4)\quad x_i = w'x_i' + w''x_i'' + \cdots + w^{(n)}x_i^{(n)} + \sum_{k=1}^{n} c_k x_i^{(k)} \qquad (i=1, \cdots, n)$$

in which

$$(5)\quad w^{(k)} = \frac{1}{\Delta}\int_0^t \left[f_1(z)\Delta_1^{(k)}(z) + f_2(z)\Delta_2^{(k)}(z) + \cdots + f_n(z)\Delta_n^{(k)}(z)\right]dz \qquad (k=1, 2, \cdots, n).$$

Now the functions $x_i^{(k)}(t)$ are of the form

$$x_i^{(k)}(t) = e^{a_k t}\varphi_i^{(k)}(t) \qquad (i, k=1, \cdots, n),$$

where $\varphi_i^{(k)}$ is a polynomial in t, of degree at most n, whose coefficients are periodic, of period T in t. It follows that $\Delta_i^{(k)}(t)$ has a similar form. For instance,

$$\Delta_1' = \begin{vmatrix} e^{a_2 t}\varphi_2'' & \cdots & e^{a_n t}\varphi_2^{(n)} \\ \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot \\ e^{a_2 t}\varphi_n'' & & e^{a_n t}\varphi_2^{(n)} \end{vmatrix} = e^{(a_2+\cdots+a_n)t} \begin{vmatrix} \varphi_2'' & \cdots & \varphi_n^{(n)} \\ \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot \\ \varphi_n'' & & \varphi_n^{(n)} \end{vmatrix}.$$

If equations (3) are those of a dynamical system,

$$a_1 + a_2 + \cdots + a_n = 0,$$

whence

$$\Delta_1' = e^{-a_1 t}\psi_1'(t),$$

$\psi_1'(t)$ being a polynominal in t with periodic coefficients, of degree at most $(n-1)$.*

In general,

$$(6)\quad \Delta_i^{(k)} = e^{-a_k t}\psi_i^{(k)}(t) \qquad (i, k=1, \cdots, n).$$

Equations (2), (a) are of the type of (3), with $n=2m$, and

$$f_i = \sum_{k=1}^{r} \frac{\partial^2 F}{\partial y_i \partial u_k}\hat{u}_k \qquad (i=1, \cdots, m),$$

* This degree is less than n, since each set of functions $\Delta_i^k(t)$ $(i=1, \cdots, n)$ is a solution of the differential system adjoint to the homogeneous system of (3).

$$f_1 = -\sum_{k=1}^{r} \frac{\partial^2 F}{\partial x_i \partial u_k} \mathfrak{u}_k \qquad (i = m+1, \cdots, n) .$$

If these functions are substituted in (5), we obtain

$$w^{(k)}(t) = \frac{1}{\Delta} \int_0^t e^{-\alpha_k z} [f_1(z)\psi_1^{(k)}(z) + \cdots + f_n(z)\psi_n^{(k)}(z)]dz$$

$$= \frac{1}{\Delta} \int_0^t e^{-\alpha_k z} \phi^{(k)}(z)dz \qquad (k = 1, \cdots, n) .$$

The function $\phi^{(k)}(z)$ is a polynominal in z with periodic coefficients, of degree at most $(n-1)$, consequently*

$$w^{(k)}(t) = e^{-\alpha_k t}\psi^{(k)}(t) , \tag{7}$$

$\psi^{(k)}(t)$ being a polynomial with periodic coefficients, of degree at most n.

Substituting these functions in (4), we obtain

$$x_i = \varphi_i'\psi' + \varphi_i''\psi'' + \cdots + \varphi_i^{(n)}\psi^{(n)} + \sum_{k=1}^{n} c^k x_i^{(k)} \quad (i = 1, \cdots, n)$$

and all the exponents different from zero, in the expression for x_i, occur in the sum $\sum_{k=1}^{n} C_k x_i^{(k)}$ and are, therefore, characteristic exponents of the homogeneous system of (α).

The integration of equations (γ) gives

$$\bar{v}_i = -\sum_{k=1}^{m} \int_0^t \frac{\partial^2 F}{\partial u_i \partial x_k} \xi_k dt - \sum_{k=1}^{m} \int_0^t \frac{\partial^2 F}{\partial u_i \partial y_k} \eta_k dt - \sum_{k=1}^{r} \int_0^t \frac{\partial^2 F}{\partial u_i \partial u_k} \mathfrak{u}_k dt ,$$

and from a discussion analogous to that already given it follows that the only exponents in this expression not of the form $\alpha = 2\pi mi/T$ (m an integer) are exponents of the functions ξ_i, η_i.

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* For each coefficient of the polynomial $\phi^k(z)$ is analytic, hence can be expanded in a Fourier series which converges absolutely and uniformly for all real values of z; and the integral can be evaluated by the term-by-term integration of these series:

$$\int_0^t e^{-\alpha_k z}\phi^{(k)}(z)dz = \sum_{s=1}^{p} \int_0^t \left\{ z^s e^{-\alpha_k z} \sum_{r=0}^{\infty} \left[A_r \cos\frac{2r\pi z}{T} + B_r \sin\frac{2r\pi z}{T} \right] \right\} dz$$

$$= \sum_{s=1}^{p} \left\{ \sum_{r=0}^{\infty} \left[A_r \int_0^t e^{-\alpha_k z} z^s \cos\frac{2r\pi z}{T} dz + B_r \int_0^t e^{-\alpha_k z} z^s \sin\frac{2r\pi z}{T} dz \right] \right\} .$$

Each integral can be expressed as a sum of terms of the same form as the integrand, of degree at most one greater in t; the numerical coefficients introduced are bounded, for all values of r, hence the series of integrated terms converges absolutely and uniformly and when expressed as a poylnomial in t, multiplied by $e^{-\alpha_k t}$, is of the form (7).

It is an immediate consequence that the characteristic exponents different from zero can be calculated either from the original system $(2; \alpha, \beta, \gamma)$ or from the reduced system $(2, \alpha)$, which are the equations of variation for the system obtained from (1) by substituting $u_i = c_i$, $\bar{u}_i = 0$, and ignoring the coördinates $v_1, \cdots, v_r$.

Another property, of importance in the calculation of the characteristic exponents, is their invariance under an analytic change of variables which does not involve the time.*

7. Reduction of the equations of motion in the neighborhood of the straight line solutions

In the following paragraphs we shall apply some of the preceding results to the study of orbits in the neighborhood of the straight line solutions, and the equilateral triangle solutions, respectively, in the problem of three bodies. The notion of *invariant relation* appears especially convenient.

If three masses m_0, m_1, m_2 have the coördinates and components of momentum

$$m_0 : (x_1, x_2, x_3, y_1, y_2, y_3) ,$$
$$m_1 : (x_4, x_5, x_6, y_4, y_5, y_6) ,$$
$$m_2 : (x_7, x_8, x_9, y_7, y_8, y_9) ,$$
$$y_i = m_0 \frac{dx_i}{dt} \qquad (i = 1, 2, 3) ,$$
$$y_i = m_1 \frac{dx_i}{dt} \qquad (i = 4, 5, 6) ,$$
$$y_i = m_2 \frac{dx_i}{dt} \qquad (i = 7, 8, 9) ,$$

and if these masses move subject to their mutual attraction according to Newton's law, the equations of motion are

$$\frac{dx_i}{dt} = \frac{\partial F}{\partial y_i} , \quad \frac{dy_i}{dt} = -\frac{\partial F}{\partial x_i} \qquad (i = 1, 2, \cdots, 9) .$$

$$(1) \qquad F = \frac{1}{2m_0} \sum_{i=1}^{3} y_i^2 + \frac{1}{2m_1} \sum_{i=4}^{6} y_i^2 + \frac{1}{2m_2} \sum_{i=7}^{9} y_i^2 - U ,$$

$$U = \frac{m_1 m_2}{r_0} + \frac{m_2 m_0}{r_1} + \frac{m_1 m_2}{r_2} .$$

* For transformations involving the time, see Poincaré, *Les Méthodes Nouvelles*, vol. 1, p. 199.

Here r_0, r_1, r_2 are the distances separating the masses (m_1, m_2), (m_2, m_0), (m_0, m_1), respectively. If the center of gravity is taken as the origin, introducing the notation

$$\lambda=\frac{m_1}{m_0+m_1}\,,\quad \mu=\frac{m_0}{m_0+m_1}\,,\quad M=m_0+m_1+m_2\,, \tag{2}$$

$$g=\frac{M}{m_2(m_0+m_1)}\,,\quad h=\frac{m_0+m_1}{m_0m_1}\,,$$

$$\bar{x}_1=x_4-x_1\,,\quad \bar{x}_2=x_5-x_2\,,\quad \bar{x}_3=x_6-x_3\,,$$

$$\bar{x}_4=gm_2x_7\,,\quad \bar{x}_5=gm_2x_8\,,\quad \bar{x}_6=gm_2x_9\,, \tag{3}$$

$$\bar{y}_i=\frac{1}{h}\frac{d\bar{x}_i}{dt} \qquad (i=1, 2, 3)\,,$$

$$\bar{y}_i=\frac{1}{g}\frac{d\bar{x}_i}{dt} \qquad (i=4, 5, 6)\,,$$

we obtain the equations of motion

$$\frac{d\bar{x}_i}{dt}=\frac{\partial F}{\partial \bar{y}_i}\,,\qquad \frac{d\bar{y}_i}{dt}=-\frac{\partial F}{\partial \bar{x}_i} \qquad (i=1, 2, \cdots, 6)\,; \tag{4}$$

$$F=\frac{h}{2}(\bar{y}_1^2+\bar{y}_2^2+\bar{y}_3^2)+\frac{g}{2}(\bar{y}_4^2+\bar{y}_5^2+\bar{y}_6)^2-U\,,$$

$$r_0^2=(\bar{x}_4-\mu\bar{x}_1)^2+(\bar{x}_5-\mu\bar{x}_2)^2+(\bar{x}_6-\mu\bar{x}_3)^2\,,$$

$$r_1^2=(\bar{x}_4+\lambda\bar{x}_1)^2+(\bar{x}_5+\lambda\bar{x}_2)^2+(\bar{x}_6+\lambda\bar{x}_3)^2\,, \tag{5}$$

$$r_2^2=\bar{x}_1^2+\bar{x}_2^2+\bar{x}_3^2\,.$$

An examination of equations (1) shows that the relations

$$x_3=x_6=x_9=y_3=y_6=y_9=0 \tag{6}$$

imply the relations

$$\frac{dx_3}{dt}=\frac{dx_6}{dt}=\frac{dx_9}{dt}=\frac{dy_3}{dt}=\frac{dy_6}{dt}=\frac{dy_9}{dt}=0\,,$$

and consequently, from the principle stated by Poincaré (§1), equations (6) form a set of invariant relations. Similarly, the equations

$$x_2=x_5=x_8=y_2=y_5=y_8=0 \tag{6'}$$

are a set of invariant relations; the relations (6) and (6′) together form a set which characterizes a movement on a fixed straight line.

Suppose p a mobile straight line through the center of gravity, taken as the origin, and let x_0', x_1', x_2' be the coördinates of m_0, m_1, m_2 respectively on p, m_1 lying between m_0 and m_2. Suppose

$$x_1' = -\alpha x_0' , \qquad x_2' = -\beta x_0' . \tag{7}$$

Then if

$$\rho_0' = |x_0'| , \quad \rho_1' = |x_1'| , \quad \rho_2' = |x_2'| ,$$

it follows that

$$r_0 = (\beta-\alpha)\rho_0' , \quad r_1 = (1+\beta)\rho_0' , \quad r_2 = (1+\alpha)\rho_0' . \tag{8}$$

If equations (6) are satisfied, and the masses m_i lie on the line p for all values of the time, while α, β are certain constants, then it can be shown* that

$$\alpha(1+\beta)^{-2} - \beta(1+\alpha)^{-2} + (\beta-\alpha)^{-2} = 0 , \tag{9}$$

$$m_0 - m_1\alpha - m_2\beta = 0 .$$

Or if

$$\beta - \alpha = (1+\alpha)\nu , \tag{10}$$

$$m_1(1+\nu)^2(1-\nu^3) - m_0\nu^2[(1+\nu)^3-1] + m_2[(1-\nu)^3-\nu^3] = 0 ,$$

which is the equation of Laplace.

If the polar coördinates are employed

$$x_1 = \rho_0' \cos\theta_0' , \qquad x_4 = \rho_1' \cos\theta_1' , \qquad x_7 = \rho_2' \cos\theta_2' ,$$

$$x_2 = \rho_0' \cos\theta_0' , \qquad x_5 = \rho_1' \cos\theta_1' , \qquad x_8 = \rho_2' \sin\theta_2'$$

and

$$F = \sum_{i=1}^{3} \frac{1}{2m_i}\left(R_i'^2 + \frac{\Theta_i'^2}{\rho_i'^2}\right) - U ,$$

equations (1) become

$$\frac{d\rho_i'}{dt} = \frac{\partial F}{\partial R_i'} , \qquad \frac{d\theta_i'}{dt} = \frac{\partial F}{\partial \Theta_i'} ,$$

$$\frac{dR_i'}{dt} = -\frac{\partial F}{\partial \rho_i'} , \qquad \frac{d\Theta_i'}{dt} = -\frac{\partial F}{\partial \theta_i'} \qquad (i = 0, 1, 2) . \tag{1′}$$

Consider the relations

$$\epsilon\pi + \theta_0' - \theta_1' = 0 , \quad \frac{d\theta_0'}{dt} - \frac{d\vartheta_1'}{dt} = 0 , \quad \rho_1' - |\alpha|\rho_0' = 0 , \quad \frac{d\rho_1'}{dt} - |\alpha|\frac{d\rho_0'}{dt} = 0 ,$$

$$\pi + \theta_0' - \theta_2' = 0 , \quad \frac{d\theta_0'}{dt} - \frac{d\theta_2'}{dt} = 0 , \quad \rho_2' - \beta\rho_0' = 0 , \quad \frac{d\rho_2'}{dt} - \beta\frac{d\rho_0'}{dt} = 0 , \tag{11}$$

* Charlier, *Die Mechanik des Himmels*, vol. 2, 1907, p. 95.

where $\epsilon=1$ if $a>0$, otherwise $\epsilon=0$. If the left hand members of (11) are zero at $t=t_0$, it can easily be shown that as a result of (11) and (1′) their derivatives are zero. Consequently equations (11) are a set of invariant relations, which characterize the straight line solutions.

Suppose

$$
\begin{aligned}
U_0' &= \frac{1}{\rho_0'}\left[\frac{m_0m_1}{(1+\alpha)^2}+\frac{m_0m_2}{(1+\beta)^2}\right]=\frac{M_0'\,m_0}{\rho_0'}\,,\\
U_1' &= \frac{\alpha^3}{|\alpha|\rho_1'}\left[\frac{-m_1m_2}{(\beta-\alpha)^2}+\frac{m_0m_1}{(1+\alpha)^2}\right]=\frac{M_1'\,m_1}{\rho_1'}\,,\\
U_2' &= \frac{\beta^2}{\rho_2'}\left[\frac{m_0m_2}{(1+\beta)^2}+\frac{m_1m_2}{(\beta-\alpha)^2}\right]=\frac{M_2'\,m_2}{\rho_2'}\,,
\end{aligned}
\tag{12}
$$

$$
F_i' = \frac{1}{2m_i}\left(R_i'^2+\frac{\Theta_i'^2}{\rho_i'^2}\right)-U_i' \qquad (i=0,\,1,\,2)\,.
$$

By means of equations (9) it can easily be shown that

$$
M_1'=|\alpha|^3M_0'\,, \qquad M_2'=\beta^3M_0'\,.
$$

If equations (11) are satisfied, equations (1′) become

$$
\begin{aligned}
&\frac{d\rho_i'}{dt}=\frac{\partial F_i'}{\partial R_i'}\,, \qquad \frac{d\theta_i'}{dt}=\frac{\partial F_i'}{\partial \Theta_i'}\,,\\
&\frac{dR_i'}{dt}=-\frac{\partial F_i'}{\partial \rho_i'} \qquad \frac{d\Theta_i'}{dt}=-\frac{\partial F_i'}{\partial \theta_i'} \qquad (i=0,\,1,\,2),
\end{aligned}
\tag{1''}
$$

as is easily verified; from these equations it follows that each mass moves in an ellipse having the center of gravity at one of the foci.

If polar coördinates corresponding to the variables (3) are employed, with the relations

$$
\bar{x}_3=\bar{x}_6=\bar{y}_3=\bar{y}_6=0 \tag{6''}
$$

which correspond to (6), these may be chosen such that $\rho_1=r_2$, and

$$
\begin{aligned}
\bar{x}_1&=\rho_1\cos\varphi_1\,, \qquad \bar{x}_4=\rho_2\cos\varphi_2\,,\\
\bar{x}_2&=\rho_1\sin\varphi_1\,, \qquad \bar{x}_5=\rho_2\sin\varphi_2\,.
\end{aligned}
\tag{13}
$$

Suppose

$$
\begin{aligned}
&\bar{m}_1=\frac{1}{h}\,, \qquad \bar{m}_2=\frac{1}{g}\,,\\
&M_1=(1+\alpha)^3M_0'\,, \qquad M_2=\left[\beta-\alpha+\mu(1+\alpha)\right]^3M_0'\,,\\
&U_1=\frac{\bar{m}_1M_1}{\rho_1}\,, \qquad U_2=\frac{\bar{m}_2M_2}{\rho_2}\,,
\end{aligned}
\tag{14}
$$

and

$$(15)\qquad \bar{F}_1=\frac{1}{2\bar{m}_1}\left(R_1^2+\frac{G_1^2}{\rho_1^2}\right)-U_1\,,\qquad \bar{F}_2=\frac{1}{2\bar{m}_2}\left(R_2^2+\frac{G_2^2}{\rho_2^2}\right)-U_2\,.$$

In terms of the variables (13) equations (4) become

$$\bar{x}_2=\bar{x}_4=\bar{y}_2=\bar{y}_4=0\,,$$

$$(16)\qquad \frac{d\rho_k}{dt}=\frac{\partial F}{\partial R_k}\,,\qquad \frac{d\varphi_k}{dt}=\frac{\partial F}{\partial G_k}\,,$$

$$\frac{dR_k}{dt}=-\frac{\partial F}{\partial \rho_k}\,,\qquad \frac{dG_k}{dt}=-\frac{\partial F}{\partial \varphi_k}\qquad (k=1,2)\,,$$

$$F=\tfrac{1}{2}\sum_{k=1}^{2}\frac{1}{\bar{m}_k}\left(R_k^2+\frac{G_k^2}{\rho_k^2}\right)-U\,.$$

Consider the relations

$$(11')\quad (a)\quad \varphi_2-\varphi_1=0\,,\qquad \rho_2-(\mu+\nu)\rho_1=0\,,$$

$$(b)\quad \frac{d\varphi_2}{dt}-\frac{d\varphi_1}{dt}=0\,,\qquad \frac{d\rho_2}{dt}-(\mu+\nu)\frac{d\rho_1}{dt}=0\,,$$

$$(c)\quad G_2-\frac{\bar{m}_2}{\bar{m}_1}(\mu+\nu)^2G_1=0\,,\quad R_2-\frac{\bar{m}_2}{\bar{m}_1}(\mu+\nu)R_1=0\,.$$

As a consequence of equations (16) equations (b) are equivalent to (c) and it can easily be shown that equations (11′) follow from (11).

It will be established, independently of (11), that equations (11′; a, c) form a set of invariant relations; as a consequence of the principle stated by Poincaré (§1) it is only necessary to show that equations (a), (c) with (16) imply that the derivatives of the left members of these equations are zero. Since equations (c) and (16) imply (b), it only remains to show that the derivatives of the left members of (c) are zero, as a consequence of (a), (c), and (16). From (5),

$$r_0^2=\rho_2^2+\mu^2\rho_1^2-2\mu\rho_1\rho_2\cos(\varphi_2-\varphi_1)\,,$$

$$r_1^2=\rho_2^2+\lambda^2\rho_1^2+2\lambda\rho_1\rho_2\cos(\varphi_2-\varphi_1)\,.$$

Consequently if $\varphi_1=\varphi_2$,

$$\frac{\partial U}{\partial\varphi_1}=\frac{\partial U}{\partial\varphi_2}=0\,,\qquad \frac{dG_1}{dt}=\frac{dG_2}{dt}=0\,.$$

It remains to show that

$$\frac{dR_2}{dt} = \frac{\bar{m}_2}{\bar{m}_1}(\mu+\nu)\frac{dR_1}{dt},$$

or from (16), that

$$\frac{G_2^2}{\bar{m}_2\rho_2^3} + \frac{\partial U}{\partial \rho_2} = \frac{\bar{m}_2}{\bar{m}_1}(\mu+\nu)\left[\frac{G_1^2}{\bar{m}_1\rho_1^3} + \frac{\partial U}{\partial \rho_1}\right].$$

But from (14), (15), (16) it is easily seen that when equations (11′; a, c) are satisfied,

$$\frac{\partial U}{\partial \rho_2} = \frac{\partial U_2}{\partial \rho_2}, \qquad \frac{\partial U}{\partial \rho_1} = \frac{\partial U_1}{\partial \rho_1}, \tag{17}$$

and from (14), (11′, a),

$$\frac{M_2}{\rho_2^2} = (\mu+\nu)\frac{M_1}{\rho_1^2};$$

from (11′; a, c)

$$\frac{G_2^2}{\bar{m}_2\rho_2^3} = \frac{\bar{m}_2}{\bar{m}_1}(\mu+\nu)\frac{G_1^2}{\bar{m}_1\rho_1^3}.$$

The equality to be established follows immediately, and the relations (11′) are therefore a set of invariant relations, which define the straight line solutions in terms of the variables (13) and the corresponding quantities R_i, G_i. When these relations are satisfied it follows from (15), (16), (17) that the equations of motion are

$$\bar{x}_3 = \bar{x}_6 = \bar{y}_3 = \bar{y}_6 = 0,$$

$$\begin{aligned} \frac{d\rho_k}{dt} &= \frac{\partial \bar{F}_k}{\partial R_k}, & \frac{d\varphi_k}{dt} &= \frac{\partial \bar{F}_k}{\partial G_k}, \\ \frac{dR_k}{dt} &= -\frac{\partial \bar{F}_k}{\partial \rho_k}, & \frac{dG_k}{dt} &= -\frac{\partial \bar{F}_k}{\partial \varphi_k} \end{aligned} \qquad (k=1, 2). \tag{16′}$$

The equations of motion can be simplified by the repeated application of a transformation

$$\begin{aligned} x_1' &= x_1 + cx_2, & x_2' &= x_2, \\ y_1' &= y_1, & y_2' &= y_2 - cy_1. \end{aligned}$$

This transformation is such that

$$x_1'dy_1' + x_2'dy_2' = x_1dy_1 + x_2dy_2$$

if the constant c is supposed given, and hence if x_1, y_1, x_2, y_2 satisfy equations

$$\frac{dx_1}{dt} = \frac{\partial H}{\partial y_1}, \qquad \frac{dx_2}{dt} = \frac{\partial H}{\partial y_2},$$
$$\frac{dy_1}{dt} = -\frac{\partial H}{\partial x_1}, \qquad \frac{dy_2}{dt} = -\frac{\partial H}{\partial x_2},$$

the equations will be canonical in terms of the new variables.

Consequently the canonical form of equations (16) is retained by the transformation

$$\varphi_1' = \varphi_1, \qquad \varphi = \varphi_2 - \varphi_1,$$
$$G_1' = G_1 + G_2, \qquad G = G_2.$$

Since $G_1+G_2=C$, it follows that $\partial F/\partial \varphi_1' = 0$, as is easily verified.*

Instead of the variables

$$\rho_1, \quad \rho_2, \quad \varphi,$$
$$R_1, \quad R_2, \quad G,$$

---

* The canonical reduction of equations (4, §7) to four degrees of freedom given by Poincaré (*Les Méthodes Nouvelles*, vol. 1, Chap. 1) can also be obtained by a repetition of this transformation, from the equations in terms of the elliptic coördinates

$$l_1, l_2, g_1, g_2, \theta_1, \theta_2,$$
$$L_1, L_2, G_1, G_2, \Theta_1, \Theta_2 \qquad (\Theta_k = G_k \cos i_k).$$

If the motion is referred to the invariable plane, from the area integrals one obtains

$$\Theta_1 + \Theta_2 = C, \quad G_1 \sin i_1 + G_2 \sin i_2 = 0, \quad \theta_2 = \theta_1.$$

Applying the transformation above, we obtain the canonical variables

$$\theta_1' = \theta_1, \qquad \theta_2' = \theta_2 - \theta_1 = 0,$$
$$C = \Theta_1 + \Theta_2, \quad \Theta_2' = \Theta_2,$$

and since $\partial F/\partial \theta_1' = \partial F/\partial \Theta_2'$, the variables θ_1', Θ_2' can be ignored, and the reduced system of Poincaré results. If the further transformation

$$h = g_1, \qquad g = g_2 - g_1,$$
$$H = G_1 + G_2 - C, \quad G = G_2$$

is made, then, from the equations above,

$$H = \frac{4G_2(C - G_2 + H)\sin^2((i_1 - i_2)/2)}{2C + H}.$$

These last canonical variables are such that a motion in the invariable plane is characterized by the single invariant relation $H=0$, if $C \neq 0$, $G_2^0(C - G_2^0) \neq 0$, and when this relation is satisfied the remaining variables define the motion with three degrees of freedom in the plane, while h is ignorable. For if $H^0=0$, $C \neq 0$, $G_2^0(C - G_2^0) \neq 0$, then $i_2 - i_1 = 2k\pi$, consequently from the area integrals we obtain

$$(G_1^0 + G_2^0)\cos i_1 = C \neq 0, \quad (G_1^0 + G_2^0)\sin i_1 = 0.$$

Hence $\sin i_1 = 0$, and the inclinations of the two osculating elliptic orbits are zero at the time $t=t^0$; whence they must be zero for all values of t, and hence $H=0$ for every value of t. Consequently

$$\frac{dH}{dt} = \frac{\partial F}{\partial h} = 0.$$

suppose

$$\rho_1' = \rho_1 , \qquad \rho_2' = \rho_2 - (\mu+\nu)\rho_1 , \qquad u_2 = \varphi ,$$

$$R_1' = R_1 + (\mu+\nu)R_2 , \; R_2' = R_2, \quad v_2 = G - \frac{\bar{m}_2(\mu+\nu)^2}{\bar{m}_1+\bar{m}_2(\mu+\nu)^2} C ,$$

and finally

$$\begin{aligned} &\rho_1'' = \rho_1' + \sigma\rho_2' , \qquad u_1 = \rho_2' , \qquad u_2 , \\ &R_1'' = R_1' , \qquad\qquad v_1 = R_2' - \sigma R_1' , \; v_2 , \end{aligned} \tag{18}$$

where

$$\sigma = \frac{\bar{m}_2(\mu+\nu)}{\bar{m}_1+\bar{m}_2(\mu+\nu)^2} .$$

In terms of the last variables the equations (11′) become

$$u_1 = v_1 = u_2 = v_2 = 0 .$$

The original variables are

$$\begin{aligned} &\rho_1 = \rho_1'' - \sigma u_1 , \qquad \rho_2 = (\mu+\nu)\rho_1'' + a u_1 , \\ &R_1 = a R_1'' - (\mu+\nu)v_1 , \quad R_2 = v_1 + \sigma R_1'' , \quad a = 1-\sigma(\mu+\nu) , \\ &\varphi = u_2 , \qquad\qquad G = v_2 + \frac{\bar{m}_2(\mu+\nu)^2 C}{\bar{m}_1+\bar{m}_2(\mu+\nu)^2} . \end{aligned} \tag{19}$$

From (16),

$$F = \frac{1}{2\bar{m}_1}\left[R_1^2 + \frac{(C-G)^2}{\rho_1^2}\right] + \frac{1}{2\bar{m}_2}\left[R_2^2 + \frac{G^2}{\rho_2^2}\right] - U ,$$

$$\begin{aligned} &r_0^2 = \rho_2^2 + \mu^2\rho_1^2 - 2\mu\rho_1\rho_2\cos\varphi , \\ &r_1^2 = \rho_2^2 + \lambda^2\rho_1^2 + 2\lambda\rho_1\rho_2\cos\varphi , \qquad\qquad r_2 = \rho_1 . \end{aligned} \tag{20}$$

After this change of variables equations (16) become

$$\begin{aligned} &\frac{d\rho_1''}{dt} = \frac{dF}{\partial R_1''} , \qquad \frac{du_1}{dt} = \frac{\partial F}{\partial v_1} , \qquad \frac{du_2}{dt} = \frac{\partial F}{\partial v_2} , \\ &\frac{dR_1''}{dt} = -\frac{\partial F}{\partial \rho_1''} , \qquad \frac{dv_1}{dt} = -\frac{\partial F}{\partial u_1} , \qquad \frac{dv_2}{dt} = -\frac{dF}{\partial u_2} . \end{aligned} \tag{21}$$

8. Reduction of the equations of motion in the neighborhood of the equilateral triangle solution

The equilateral triangle solutions and the solutions in a certain neighborhood of them can be discussed by methods analogous to those employed in §7. Only plane movements will be treated here.

Suppose the equations of motion in the form (16, §7),

$$(1)\qquad \frac{d\rho_k}{dt}=\frac{\partial F}{\partial R_k},\quad \frac{d\varphi_k}{dt}=\frac{\partial F}{\partial G_k},$$
$$\frac{dR_k}{dt}=-\frac{\partial F}{\partial \rho_k},\quad \frac{dG_k}{dt}=-\frac{\partial F}{\partial \varphi_k}\qquad (k=1,2),$$
$$F=\frac{1}{2}\sum_{k=1}^{2}\frac{1}{\bar m_k}\left(R_k^2+\frac{G_k^2}{\rho_k^2}\right)-U,$$

and consider the relations

$$(2)\qquad (a)\qquad \rho_2-\gamma\rho_1=0,\quad \varphi_2-\varphi_1-\omega=0,$$
$$(b)\qquad \frac{d\rho_2}{dt}-\gamma\frac{d\rho_1}{dt}=0,\quad \frac{d\varphi_2}{dt}-\frac{d\varphi_1}{dt}=0,$$
$$(c)\qquad R_2-\gamma\frac{\bar m_2}{\bar m_1}R_1=0,\quad G_2-\frac{\bar m_2}{\bar m_1}\gamma^2G_1=0,$$

where

$$(3)\qquad \gamma=\frac{\sqrt{m_0^2+m_0m_1+m_1^2}}{m_0+m_1},\qquad \cos\omega=\frac{m_0-m_1}{2\sqrt{m_0^2+m_0m_1+m_1^2}}.$$

Now suppose

$$(4)\qquad \rho_1=r_2,\quad \bar M_1=M=m_0+m_1+m_2,\quad \bar M_2=\gamma^3M,$$
$$U_1=\frac{\bar m_1\bar M_1}{\rho_1},\qquad U_2=\frac{\bar m_2\bar M_2}{\rho_2}.$$

If equations (2, a) are satisfied, one obtains easily

$$r_0=r_1=r_2,\qquad \bar U_1+\bar U_2=U,$$
$$(5)\qquad \frac{\partial U}{\partial\rho_1}=\frac{\partial\bar U_1}{\partial\rho_1},\quad \frac{\partial U}{\partial\rho_2}=\frac{\partial\bar U_2}{\partial\rho_2},\qquad \frac{\partial U}{\partial\varphi_1}=\frac{\partial U}{\partial\varphi_2}=0.$$

To show that equations (2; a, c) form a set of invariant relations, we shall show that (1) and (2; a, c) imply that the derivatives of the left hand

members of (2; a, c) are zero. Since (1) and (2, c) imply (2, b), and the equations (5) imply the relation

$$\frac{dG_1}{dt} = \frac{dG_2}{dt} = 0 ,$$

it only remains to show that

$$-\frac{\partial F}{\partial \rho_2} + \gamma \frac{\bar{m}_2}{\bar{m}_1} \frac{\partial F}{\partial \rho_1} = 0 ,$$

or

$$\frac{1}{\bar{m}_2} \frac{G_2^2}{\rho_2^3} + \frac{\partial \bar{U}_2}{\partial \rho_2} = \gamma \frac{\bar{m}_2}{\bar{m}_1} \left[\frac{G_1^2}{\bar{m}_1 \rho_1^3} + \frac{\partial \bar{U}_1}{\partial \rho_1} \right] .$$

From (2; a, c),

$$\frac{1}{\bar{m}_2} \frac{G_2^2}{\rho_2^3} = \gamma \frac{\bar{m}_2}{\bar{m}_1} \frac{G_1^2}{\bar{m}_1 \rho_1^3} ,$$

and from (4)

$$\frac{\partial \bar{U}_2}{\partial \rho_2} = \gamma \frac{\bar{m}_2}{\bar{m}_1} \frac{\partial \bar{U}_1}{\partial \rho_1} .$$

Hence, as a consequence of the principal stated by Poincaré, the relations (2; a, c) form a set of invariant relations. When these relations are satisfied, it follows from the discussion just given that if

$$F_1 = \frac{1}{2\bar{m}_1} \left(R_1^2 + \frac{G_1^2}{\rho_1^2} \right) - \bar{U}_1 ,$$

$$F_2 = \frac{1}{2\bar{m}_2} \left(R_2^2 + \frac{G_2^2}{\rho_2^2} \right) - \bar{U}_2 ,$$

the equations (1) become

$$(6) \qquad \begin{aligned} \frac{d\rho_1}{dt} &= \frac{\partial F_i}{\partial R_i} , & \frac{d\varphi_i}{dt} &= \frac{\partial F_i}{\partial G_i} , \\ \frac{dR_i}{dt} &= -\frac{\partial F_i}{\partial \rho_i} , & \frac{dG_i}{dt} &= -\frac{\partial F_i}{\partial \varphi_i} \end{aligned} \qquad (i=1,2) .$$

From these equations the elliptic form of the relative orbits is evident. As in the preceding paragraph, if the change of variables

$$(7) \qquad \begin{aligned} \varphi_1' &= \varphi_1 , & \varphi &= \varphi_2 - \varphi_1 , \\ C &= G_1 + G_2 , & G &= G_2 \end{aligned}$$

is made, $C=\text{const.}$, and the motion is defined by the canonical variables

$$\begin{matrix} \rho_1 , & \rho_2 , & \varphi , \\ R_1 , & R_2 , & G . \end{matrix}$$

Suppose the canonical change of variables made,

$$\begin{matrix} \rho_1'=\rho_1 , & \rho_2'=\rho_2-\gamma\rho_1 , & u_2=\varphi-\omega , \\ R_1'=R_1+\gamma R_2 , & R_2'=R_2 , & v_2=G-\sigma' C \end{matrix} \qquad \left(\sigma'=\frac{\bar{m}_2\gamma^2}{\bar{m}_1+\bar{m}_2\gamma^2}\right),$$

and finally

$$(8) \qquad \begin{matrix} \rho_1''=\rho_1'+mu_1 , & u_1=\rho_2-\gamma\rho_1 , & u_2=\varphi-\omega , \\ R_1''=R_1' , & v_1=R_2'-mR_1' , & v_2=G-\sigma' C , \end{matrix} \qquad m=\frac{\bar{m}_2\gamma}{\bar{m}_1+\gamma^2\bar{m}_2} .$$

The relations (2; a, c) become

$$(2') \qquad u_1=v_1=u_2=v_2=0 ,$$

and the equations of motion

$$(1'') \qquad \begin{matrix} \dfrac{d\rho_1''}{dt}=\dfrac{\partial F}{\partial R_1''} , & \dfrac{du_1}{dt}=\dfrac{\partial F}{\partial v_1} , & \dfrac{du_2}{dt}=\dfrac{\partial F}{\partial v_2} , \\ \dfrac{dR_1''}{dt}=-\dfrac{\partial F}{\partial \rho_1''} , & \dfrac{dv_1}{dt}=-\dfrac{\partial F}{\partial u_1} , & \dfrac{dv_2}{dt}=-\dfrac{\partial F}{\partial u_2} . \end{matrix}$$

The original variables are defined by the equations

$$\begin{matrix} R_1=(1-m\gamma)R_1''-\gamma v_1 , & \rho_1=\rho_1''-mu_1 , & \varphi=\omega+u_2 , \\ R_2=mR_1''+v_1 , & \rho_2=\gamma\rho_1''+(1-m\gamma)u_1 , & G=\sigma' C+v_2 . \end{matrix}$$

9. Orbits in the neighborhood of the straight line solutions

The behavior of the solutions of equations (21, §7), in the neighborhood of a straight line solution depends upon the properties of the equations of variation, corresponding to a solution for which $u_1=v_1=u_2=v_2=0$.

Since

$$\begin{aligned} r_0^2&=\rho_2^2+\mu^2\rho_1^2-2\mu\rho_1\rho_2+2\mu\rho_1\rho_2(1-\cos\varphi) \\ &=(\rho_2-\mu\rho_1)^2\left[1+\frac{4\mu\rho_1\rho_2}{(\rho_2-\mu\rho_1)^2}\sin^2\frac{\varphi}{2}\right] , \end{aligned}$$

we obtain

$$\frac{1}{r_0}=\frac{1}{\rho_2-\mu\rho_1}\left[1-\frac{2\mu\rho_1\rho_2}{(\rho_2-\mu\rho_1)^2}\sin^2\frac{\varphi}{2}+\frac{1\cdot3}{2\cdot4}\left(\frac{4\mu\rho_1\rho_2}{(\rho_2-\mu\rho_1)^2}\sin^2\frac{\varphi}{2}\right)^2+\cdots\right]$$

and, similarly,

$$\frac{1}{r_1}=\frac{1}{\rho_2+\lambda\rho_1}\left[1+\frac{2\lambda\rho_1\rho_2}{(\rho_2+\lambda\rho_1)^2}\sin^2\frac{\varphi}{2}+\frac{1\cdot3}{2\cdot4}\left(\frac{4\lambda\rho_1\rho_2}{(\rho_2+\lambda\rho_1)^2}\sin^2\frac{\varphi}{2}\right)^2+\cdots\right].$$

Consequently

$$U=\frac{m_1m_2}{\rho_2-\mu\rho_1}+\frac{m_2m_0}{\rho_2+\lambda\rho_1}+\frac{m_0m_1}{\rho_1}+2\rho_1\rho_2\left[\frac{\lambda m_2m_0}{(\rho_2+\lambda\rho_1)^3}-\frac{\mu m_1m_2}{(\rho_2-\mu\rho_1)^3}\right]\sin^2\frac{u_2}{2}+U_3,$$

where U_3 is a power series in $\sin^2(u_2/2)$ of at least the second degree in $\sin^2(u_2/2)$. It follows immediately that

$$\frac{\partial^2F}{\partial u_2^2}=-\rho_1\rho_2\left[\frac{\lambda m_2m_0}{(\rho_2+\lambda\rho_1)^3}-\frac{\mu m_1m_2}{(\rho_2-\mu\rho_1)^3}\right]+u_2(\cdots),$$

and when $u_1=v_1=u_2=v_2=0$,

$$\frac{\partial^2F}{\partial u_2\partial u_1}=\frac{\partial^2F}{\partial u_2\partial v_1}=\frac{\partial^2F}{\partial u_2\partial v_2}=0.$$

If

$$A_{11}=\frac{\partial^2F}{\partial u_1^2},\quad A_{22}=\frac{\partial^2F}{\partial v_1^2},\quad B_{11}=\frac{\partial^2F}{\partial u_2^2},$$

$$B_{12}=B_{21}=\frac{\partial^2F}{\partial u_1\partial v_2},\quad B_{22}=\frac{\partial^2F}{\partial v_2^2},$$

all the remaining second order partial derivatives of F are zero under the conditions above; if e is the excentricity of the straight line solutions, a_1 the semi-major axis of the ellipse whose radius vector is ρ_1, we obtain

$$A_{11}=\frac{3\bar{m}_1a_1(1-e^2)\sigma M_1}{(\mu+\nu)\rho_1^4}-\frac{2}{\rho_1^3}\left[\frac{m_1m_2(1-\sigma\nu)^2}{\nu^3}+\frac{m_2m_0(a-\lambda\sigma)^2}{(1+\nu)^3}+m_0m_1\sigma^2\right],$$

$$A_{22}=\frac{\mu+\nu}{\bar{m}_1\sigma},\qquad B_{11}=\frac{(\mu+\nu)m_0m_1m_2}{(m_0+m_1)\rho_1}\left[\frac{1}{\nu^3}-\frac{1}{(1+\nu)^3}\right],$$

$$B_{12}=\frac{-2\sqrt{M_1a_1(1-e^2)}}{(\mu+\nu)\rho_1^2},\qquad B_{22}=\frac{1}{\bar{m}_1\sigma(\mu+\nu)\rho_1^2}.$$

Since $u_i=v_i=0$ form a set of invariant relations, the equations of variation are necessarily in the form of §2; consequently, if $\rho_1''=\rho_1^0+\xi$, $R_1''=R_1^0+\eta$,

$$\frac{d\xi}{dt}=\frac{\partial^2 F}{\partial R_1''^2}\eta\ , \qquad \frac{d\eta}{dt}=-\frac{\partial^2 F}{\partial \rho_1''^2}\xi\ ,$$

$$\frac{d\bar{u}_1}{dt}= \qquad\qquad A_{22}\bar{v}_1\ ,$$

$$\frac{d\bar{u}_2}{dt}= B_{12}\bar{u}_1 \qquad\qquad +B_{22}\bar{v}_2\ ,$$

$$\frac{d\bar{v}_1}{dt}=-A_{11}\bar{u}_1 \qquad\qquad -B_{12}\bar{v}_2\ ,$$

$$\frac{d\bar{v}_2}{dt}= \qquad -B_{11}\bar{u}_2\ .$$

If $e=0$ for the straight line solutions, the characteristic exponents different from zero are obtained from the equation

$$\Delta(S)=\begin{vmatrix} -s & 0 & A_{22} & 0 \\ B_{12} & -s & 0 & B_{22} \\ -A_{11} & 0 & -s & -B_{12} \\ 0 & -B_{11} & 0 & -s \end{vmatrix}=0\ ,$$

or

$$s^4+[A_{11}A_{22}+B_{11}B_{22}]s^2+B_{11}A_{22}[A_{11}B_{22}-B_{12}^2]=0\ .$$

Now $B_{11}A_{22}>0$, $B_{22}>0$, and if A_{11}' is the first term of the expression for A_{11}, it is found immediately that

$$A_{11}'B_{22}-B_{12}^2=\frac{-M_1}{(\mu+\nu)^2a_1^5}\ ;$$

consequently the product of the two roots s_1^2, s_2^2 of the equation above is negative. Hence two of the characteristic exponents are real, and two are pure imaginaries. It is easy to find particular cases in which none of these exponents are multiples of ni if $T=2\pi/n$ is the period of the generating solution.

10. Orbits in the neighborhood of the equilateral triangle solutions

As in the preceding paragraph, we shall set up the equations of variation for the equilateral triangle solution considered.

From the relations

$$r_0^2=\rho_2^2+\mu^2\rho_1^2-2\mu\rho_1\rho_2\cos(\omega+u_2)\,,$$

$$r_1^2=\rho_2^2+\lambda^2\rho_1^2+2\lambda\rho_1\rho_2\cos(\omega+u_2)\,,$$

$$F=\frac{1}{2\bar{m}_1}\left(R_1^2+\frac{(C-G)^2}{\rho_1^2}\right)+\frac{1}{2\bar{m}_2}\left(R_2^2+\frac{G^2}{\rho_2^2}\right)-U\,,$$

$$U=\frac{m_1m_2}{r_0}+\frac{m_2m_0}{r_1}+\frac{m_0m_1}{\rho_1}$$

we obtain easily the expansion of F in powers of u_i, v_i. From the fact that the relations $u_i=v_i=0$ form a set of invariant relations it follows immediately that

$$\left[\frac{\partial F}{\partial u_i}\right]_0=\left[\frac{\partial F}{\partial v_i}\right]_0=0 \qquad (i=1,\,2)$$

if these derivatives are computed for an equilateral triangle solution. If

$$A_{11}=\frac{\partial^2F}{\partial u_1^2}\,,\qquad A_{12}=\frac{\partial^2F}{\partial u_1\partial u_2}\,,\qquad B_{12}=\frac{\partial^2F}{\partial u_1\partial v_2}\,,$$

$$A_{22}=\frac{\partial^2F}{\partial u_2^2}\,,\qquad B_{11}=\frac{\partial^2F}{\partial v_1^2}\,,\qquad B_{22}=\frac{\partial^2F}{\partial v_2^2}\,,$$

the values of R_1'', ρ_1'' for an equilateral triangle solution being substituted in these derivatives, we obtain

$$A_{11}=\frac{3G_1^2m^2}{m_1\rho_1^4}+\frac{3G_2^2(1-m\gamma)^2}{\bar{m}_2\rho_2^4}+\frac{m_1m_2}{\rho_1^3}[m^2\mu^2+(1-m\gamma)^2]$$
$$+\frac{m_2m_0}{\rho_1^3}[m^2\lambda^2+(1-m\gamma)^2]-\frac{3m_1m_2}{\rho_1^5}[m\mu R-(1-m\gamma)\bar{U}]^2$$
$$-\frac{3m_2m_0}{\rho_1^5}[m\lambda S-(1-m\gamma)V]^2-\frac{2m^2m_0m_1}{\rho_1^3}\,,$$

$$A_{12}=\frac{3m_0m_1m_2\gamma\sin\omega\cos\omega}{(m_0+m_1)\rho_1^2}\,,\qquad B_{12}=\frac{-2mG_1}{\bar{m}_1\rho_1^3}-\frac{2(1-m\gamma)G_2}{\bar{m}_2\rho_2^3}\,,$$

$$A_{22}=\frac{-3m_0m_1m_2\gamma^2\sin^2\omega}{(m_0+m_1)\rho_1}\,,$$

$$B_{11}=\frac{\gamma^2}{\bar{m}_1}+\frac{1}{\bar{m}_2}\,,\qquad B_{22}=\frac{B_{11}}{\rho_1^2\gamma^2}\,,$$

where

$$\bar{U}=\rho_2-\mu\rho_1\cos\omega\,,\qquad V=\rho_2+\lambda\rho_1\cos\omega\,,$$
$$R=\mu\rho_1-\rho_2\cos\omega\,,\qquad S=\lambda\rho_1+\rho_2\cos\omega\,.$$

It is seen that all the remaining partial derivatives of the second order vanish for an equilateral triangle solution. The equations of variation for these solutions become, if

$$\rho_1''=\rho_1^0+\xi\,,\qquad R_1''=R_1^0+\eta\,,$$
$$\frac{d\bar{\xi}}{dt}=\frac{\partial^2 F}{\partial R_1''^2}\bar{\eta}\,,\qquad \frac{d\bar{\eta}}{dt}=-\frac{\partial^2 F}{\partial \rho_1''^2}\bar{\xi}\,,$$
$$\frac{d\bar{u}_1}{dt}=\qquad\qquad B_{11}\bar{v}_1\,,$$
$$\frac{d\bar{u}_2}{dt}=B_{11}\bar{u}_1\qquad\qquad +B_{22}\bar{v}_2\,,$$
$$\frac{d\bar{v}_1}{dt}=-A_{11}\bar{u}_1-A_{12}\bar{u}_2\qquad -B_{12}\bar{v}_2\,,$$
$$\frac{d\bar{v}_2}{dt}=-A_{12}\bar{u}_1-A_{22}\bar{u}_2\,.$$

If the eccentricity of the generating orbit is zero, the coefficients A_{ij}, B_{ij} are constant, and the characteristic exponents of the given orbit satisfy the equation

$$\Delta'(s)=\begin{vmatrix} -s & 0 & B_{11} & 0\\ B_{12} & -s & 0 & B_{22}\\ -A_{11} & -A_{12} & -s & -B_{12}\\ -A_{12} & -A_{22} & 0 & -s\end{vmatrix}=0\,,$$

or

$$s^4+[A_{11}B_{11}+A_{22}B_{22}]s^2+A_{22}B_{11}[A_{11}B_{22}-B_{12}^2]-B_{11}B_{22}A_{12}^2=0\,.$$

This equation becomes, after some reductions,*

$$s^4+\frac{m_0+m_1-m_2}{a^3}s^2+\frac{27}{4a^6}(m_0m_1+m_1m_2+m_2m_0)=0\,.$$

* For a different deduction of a similar equation, see D. Buchanan, these Transactions, loc. cit., p. 415, equation (11).

11. Orbits asymptotic to the straight line and equilateral triangle solutions, respectively

If in equations (1'', §8) we put $\rho_1'' = \rho_1^0 + \xi$, $R_1'' = R_1^0 + \eta$, and expand F in powers of ξ and η, where ρ_1^0, R_1^0 satisfy the equations of the straight line solution, F is of the second degree with respect to η, but a power series in ξ. If the equations of variation for $\bar{\xi}$, $\bar{\eta}$ are integrated, the solutions are periodic of period $2\pi/n$, and accordingly the characteristic exponents are $\pm ni$.

From the theory of linear equations with periodic coefficients it follows that by means of one linear transformation with periodic coefficients the equations in $\bar{\xi}$, $\bar{\eta}$ can be reduced to a system with constant coefficients, and the equations in $\bar{u}_i$, $\bar{v}_i$ can be similarly reduced by means of a linear transformation with coefficients of period $2\pi/n$ in t. These equations can in fact be given the form

$$
\begin{aligned}
&\frac{d\bar{\xi}}{dt} = ni\bar{\xi}', \qquad \frac{d\eta'}{dt} = -ni\bar{\eta}', \\
&\frac{d\bar{u}_1'}{dt} = s_1\bar{u}_1', \qquad d\bar{u}_2' = s_2\bar{u}_2', \\
&\frac{d\bar{v}_1'}{dt} = s_3\bar{v}_1', \qquad \frac{d\bar{v}_2'}{dt} = s_4\bar{v}_2'.
\end{aligned} \tag{1}
$$

If the variables ξ, η, u_i, v_i are subjected to the same linear transformation with periodic coefficients, the equations of motion (21, §7) become, after this transformation,

$$
\begin{aligned}
&\frac{d\xi'}{dt} = ni\xi + \varphi_2'(\xi, \eta, u_1', u_2', v_1', v_2', t), \\
&\frac{d\eta'}{dt} = -ni\eta' + \varphi_2'', \\
&\frac{du_1'}{dt} = s_1u_1' + \varphi_2''', \\
&\frac{du_2'}{dt} = s_2u_2' + \varphi_2^{(4)}, \\
&\frac{dv_1'}{dt} = s_3v_1' + \varphi_2^{(5)}, \\
&\frac{dv_2'}{dt} = s_4v_2' + \varphi_2^{(6)},
\end{aligned} \tag{2}
$$

where $\varphi_2', \cdots, \varphi_2^{(6)}$ are of at least the second degree with respect to the variables ξ', η', u_i', v_i' and periodic of period $T=2\pi/n$ with respect to t. We can at once apply the method of Picard to determine solutions of (2) developed in powers of $e^{s_1 t}$ if s_1 has its real part negative, and it has been seen that there is just one characteristic exponent having this property, the others being pure imaginaries or positive. Hence no sum of the form

$$p_1 s_1 + \gamma' i - s_h \qquad (h=1, 2, 3)$$

can vanish, if $p_1 \geqq 2$. The variable θ of Picard* is here e^t; the application of Picard's method gives one solution developed in powers of $e^{s_1 t}$ which is such that ξ', η', u_i', v_i' approach zero as $t \to \infty$; by choosing the other real exponent one obtains in precisely the same manner a solution approaching the straight line solution as $t \to -\infty$.

The equations of motion for the equilateral triangle solution can be transformed into the form (2) in precisely the same manner, and the method of Picard can again be applied.

In this case two exponents s_1, s_2 have their real parts negative, hence, for certain ranges of the masses and e, solutions can be obtained developed in powers of $e^{s_1 t}$, $e^{s_2 t}$ which approach the equilateral triangle solutions as $t \to \infty$. In the same manner another family of solutions can be obtained, which approach the equilateral triangle solutions as $t \to -\infty$; the variables are here developed in powers of $e^{s_1 t}$, $e^{s_2 t}$, the coefficients being again periodic in t of period $T=2\pi/n$.

Solutions asymptotic to the straight line solutions for which the excentricity is zero have been discussed by D. Buchanan† for arbitrary masses; it follows from the results of §9 that of the four cases mentioned by him only Case I is possible.

While the series obtained in this manner are not real, the arbitrary constants can be chosen as conjugate imaginaries, and the periodic functions which enter into the transformations can be chosen in conjugate pairs in such a manner that the original variables are real functions of t. The calculation of these series is outlined very completely in the two papers of D. Buchanan, following another method, if $e=0$ for the generating solution; with the variables above and the method of Picard the corresponding asymptotic solutions can be obtained for e arbitrary within certain limits.

* *Traité d' Analyse*, vol. 3, Chap. 8 (2d edition).

† Rendiconti del Circolo Matematico Palermo, vol. 45 (1921), pp. 1-25.

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ON CERTAIN FAMILIES OF ORBITS WITH ARBITRARY MASSES IN THE PROBLEM OF THREE BODIES (SECOND PAPER)*

BY

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It is the object of this paper to obtain theorems concerning the stability of the straight line solutions, and equilateral triangle solutions, respectively, in the problem of three bodies by means of the theorems and calculations of two preceding papers.† It is shown that the generalized theorems of Bohl can be applied to a neighborhood of the straight line solutions, with arbitrary masses, and to a neighborhood of the equilateral triangle solutions, if the masses are such that the characteristic exponents of the generating orbit are not all pure imaginaries. The mutual distances of the three masses are assumed constant on the generating orbit, in both cases.

1. The equations of motion

From the classical reduction of the equations of motion of the problem of three bodies these are obtained in the form

$$\frac{d\bar{x}_i}{dt}=\frac{\partial F}{\partial \bar{y}_i}, \quad \frac{d\bar{y}_i}{dt}=-\frac{\partial F}{\partial \bar{x}_i} \qquad (i=1, 2, \cdots, 6),$$

$$F=\frac{h}{2}\sum_{i=1}^{3}\bar{y}_i^2+\frac{g}{2}\sum_{i=4}^{6}\bar{y}_i^2-U, \qquad U=\frac{m_1m_2}{r_0}+\frac{m_2m_0}{r_1}+\frac{m_0m_1}{r_2},$$

$$r_0^2=(\bar{x}_4-\mu\bar{x}_1)^2+(\bar{x}_5-\mu\bar{x}_2)^2+(\bar{x}_6-\mu\bar{x}_3)^2,$$

$$r_1^2=(\bar{x}_4+\lambda\bar{x}_1)^2+(\bar{x}_5+\lambda\bar{x}_2)^2+(\bar{x}_6+\lambda\bar{x}_3)^2, \tag{1}$$

$$r_2^2=\bar{x}_1^2+\bar{x}_2^2+\bar{x}_3^2,$$

$$\lambda=\frac{m_1}{m_0+m_1}, \qquad \mu=\frac{m_0}{m_0+m_1},$$

$$g=\frac{m_0+m_1+m_2}{m_2(m_0+m_1)}, \quad h=\frac{m_0+m_1}{m_0m_1}.$$

* Presented to the Society, December 30, 1924; received by the editors in April, 1925.

† (B) *Generalization of certain theorems of Bohl*, American Journal of Mathematics, vol. 47 (1925); (A) *On certain families of orbits with arbitrary masses in the problem of three bodies*, these Transactions, vol. 28 (1926), pp. 74-108.

f the transformation to new coördinates is made,

$$\text{(2)}\qquad \begin{aligned} &\bar{x}_1=\rho_1\cos\varphi_1 \qquad \bar{x}_4=\rho_2\cos\varphi_2\ , \qquad \bar{m}_1=\frac{1}{h}\ ,\\ &\bar{x}_2=\rho_1\sin\varphi_1 \qquad \bar{x}_5=\rho_2\sin\varphi_2\ , \qquad \bar{m}_2=\frac{1}{g}\ , \end{aligned}$$

equations (1) become

$$\text{(3)}\qquad \begin{aligned} &\frac{d\rho_k}{dt}=\frac{\partial F}{\partial R_k}\ , \quad \frac{dR_k}{dt}=-\frac{\partial F}{\partial \rho_k}\ ,\\ &\frac{d\varphi_k}{dt}=\frac{\partial F}{\partial G_k}\ , \quad \frac{dG_k}{dt}=-\frac{\partial F}{\partial \varphi_k} && (k=1,\,2)\ ,\\ &\frac{d\bar{x}_i}{dt}=\frac{\partial F}{\partial \bar{y}_i}\ , \quad \frac{d\bar{y}_i}{dt}=-\frac{\partial F}{\partial \bar{x}_i} && (i=3,\,6)\ , \end{aligned}$$

where

$$F=\frac{1}{2\bar{m}_1}\left(R_1^2+\frac{G_1^2}{\rho_1^2}+\bar{y}_3^2\right)+\frac{1}{2\bar{m}_2}\left(R_2^2+\frac{G_2^2}{\rho_2^2}+\bar{y}_6^2\right)-U\ ,$$

$$\text{(4)}\qquad \begin{aligned} &r_0^2=\rho_2^2+\mu^2\rho_1^2-2\mu\rho_1\rho_2\cos(\varphi_2-\varphi_1)+(\bar{x}_6-\mu\bar{x}_3)^2\ ,\\ &r_1^2=\rho_2^2+\lambda^2\rho_1^2+2\lambda\rho_1\rho_2\cos(\varphi_2-\varphi_1)+(\bar{x}_6+\lambda\bar{x}_3)^2\ ,\\ &r_2^2=\rho_1^2+\bar{x}_3^2\ . \end{aligned}$$

On account of the area integral

$$(\bar{x}_1\bar{y}_2-\bar{x}_2\bar{y}_1)+(\bar{x}_4\bar{y}_5-\bar{x}_5\bar{y}_4)=\text{const.}\ ,$$

we have

$$G_1+G_2=C\ .$$

If the transformation

$$\text{(5)}\qquad \begin{aligned} &\varphi_1'=\varphi_1, && \varphi=\varphi_2-\varphi_1\ ,\\ &C=G_1+G_2, && G=G_2 \end{aligned}$$

is made, the canonical form of the equations is retained, and the new equations are

$$\text{(6)}\qquad \begin{aligned} &\frac{d\varphi}{dt}=\frac{\partial F}{\partial G}\ , \quad \frac{dG}{dt}=-\frac{\partial F}{\partial \varphi}\ , \quad \frac{d\varphi_1'}{dt}=\frac{\partial F}{\partial C}\ , \quad \frac{dC}{dt}=0\ ,\\ &\frac{d\rho_i}{dt}=\frac{\partial F}{\partial R_i}\ , \quad \frac{dR_i}{dt}=-\frac{\partial F}{\partial \rho_i} && (i=1,\,2)\ ,\\ &\frac{d\bar{x}_i}{dt}=\frac{\partial F}{\partial \bar{y}_i}\ , \quad \frac{d\bar{y}_i}{dt}=-\frac{\partial F}{\partial \bar{x}_i} && (i=3,\,6)\ , \end{aligned}$$

where

$$F=\frac{1}{2\bar{m}_1}\left[R_1^2+\frac{(C-G)^2}{\rho_1^2}+\bar{y}_1^2\right]+\frac{1}{2\bar{m}_2}\left[R_2^2+\frac{G^2}{\rho_2^2}+\bar{y}_2^2\right]-U.$$

We have also

$$r_0^2=\bar{r}_0^2+(\bar{x}_6-\mu\bar{x}_3)^2\,,\qquad \bar{r}_0^2=\rho_2^2+\mu^2\rho_1^2-2\mu\rho_1\rho_2\cos\varphi\,,$$

$$r_0^2=\bar{r}_1^2+(\bar{x}_6+\lambda\bar{x}_3)^2\,,\qquad \bar{r}_1^2=\rho_2^2+\lambda^2\rho_1^2+2\lambda\rho_1\rho_2\cos\varphi\,, \tag{7}$$

$$r_2^2=\rho_1^2+\bar{x}_3^2\,.$$

For later computations it will be necessary to expand the function F in powers of $\bar{x}_3$, $\bar{x}_6$. We have

$$\begin{aligned}\frac{1}{r_0}&=\frac{1}{\bar{r}_0}\left[1+\left(\frac{\bar{x}_6-\mu\bar{x}_3}{\bar{r}_0}\right)^2\right]^{-\frac{1}{2}}\\&=\frac{1}{\bar{r}_0}\left[1-\frac{1}{2}\left(\frac{\bar{x}_6-\mu\bar{x}_3}{\bar{r}_0}\right)^2+\frac{1\cdot3}{2\cdot4}\left(\frac{\bar{x}_6-\mu\bar{x}_3}{\bar{r}_0}\right)^4+\cdots\right],\\ \frac{1}{r_1}&=\frac{1}{\bar{r}_1}\left[1+\left(\frac{\bar{x}_6+\lambda\bar{x}_3}{\bar{r}_1}\right)^2\right]^{-\frac{1}{2}}\\&=\frac{1}{\bar{r}_1}\left[1-\frac{1}{2}\left(\frac{\bar{x}_6+\lambda\bar{x}_3}{\bar{r}_1}\right)^2+\frac{1\cdot3}{2\cdot4}\left(\frac{\bar{x}_6+\lambda\bar{x}_3}{\bar{r}_1}\right)^4+\cdots\right],\\ \frac{1}{r_2}&=\frac{1}{\rho_1}\left[1+\left(\frac{\bar{x}_3}{\rho_1}\right)^2\right]^{-\frac{1}{2}}\\&=\frac{1}{\rho_1}\left[1-\frac{1}{2}\left(\frac{\bar{x}_3}{\rho_1}\right)^2+\frac{1\cdot3}{2\cdot4}\left(\frac{\bar{x}_3}{\rho_1}\right)^4+\cdots\right].\end{aligned} \tag{8}$$

Consequently,

$$U=\frac{m_1m_2}{\bar{r}_0}+\frac{m_2m_0}{\bar{r}_1}+\frac{m_0m_1}{\bar{r}_2}-\frac{1}{2}\left[m_1m_2\frac{(\bar{x}_6-\mu\bar{x}_3)^2}{\bar{r}_0^3}+m_2m_0\frac{(\bar{x}_6+\lambda\bar{x}_3)^2}{\bar{r}_1^3}+m_0m_1\frac{\bar{x}_3^2}{\rho_1^3}\right]+\bar{U}\,,$$

and

$$F=F_1+F_2+F_3\,,$$

$$\begin{aligned}F_1&=\frac{1}{2\bar{m}_1}\left[R_1^2+\frac{(C-G)^2}{\rho_1^2}\right]+\frac{1}{2\bar{m}_2}\left[R_2^2+\frac{G^2}{\rho_2^2}\right]-\frac{m_1m_2}{\bar{r}_0}-\frac{m_2m_0}{\bar{r}_1}-\frac{m_0m_1}{\bar{r}_2}\,,\\ F_2&=\frac{\bar{y}_2^2}{2\bar{m}_1}+\frac{\bar{y}_6^2}{2\bar{m}_2}+\frac{m_1m_2}{2}\frac{(\bar{x}_6-\mu\bar{x}_3)^2}{\bar{r}_0^3}+\frac{m_2m_0}{2}\frac{(\bar{x}_6+\lambda\bar{x}_3)^2}{\bar{r}_1^3}+\frac{m_0m_1}{2}\frac{\bar{x}_3^2}{\rho_1^3}\,,\end{aligned} \tag{9}$$

and F_3 contains only powers of $\bar{x}_3$, $\bar{x}_4$, not lower than the third.

Since the invariant relations

$$\bar{x}_3=\bar{x}_4=\bar{y}_3=\bar{y}_4=0 \tag{10}$$

define a movement in the plane, these may be assumed satisfied in the equations above if it is desired to study the orbits in this plane.

2. Orbits in the neighborhood of the straight line solutions

In (A), §8, it was shown that if the equations (10, §1), are satisfied, linear combinations of the remaining variables can be so chosen that the equations of motion retain the canonical form, while the straight line solutions are characterized by the invariant relations*

$$u_1=u_2=v_1=v_2=0\ . \tag{1}$$

When these relations are satisfied, the function F becomes, if $\rho_1''=\rho_1$, and

$$m'=\bar{m}_1+\bar{m}_2(\mu+\nu)^2,\quad M'm'=m_0m_1+\frac{m_0m_2}{1+\nu}+\frac{m_1m_2}{\nu}\ ,$$

$$F^0=\frac{1}{2m'}\left[R_1''^2+\frac{C^2}{\rho_1''^2}\right]-\frac{M'm'}{\rho_1''}\ . \tag{2}$$

Equations (1) being satisfied, the remaining equations

$$\frac{d\rho_1''}{dt}=\frac{1}{m'}R_1'',\qquad \frac{dR_1''}{dt}=-\frac{\partial F^0}{\partial\rho_1''}$$

are satisfied by $R_1''=0$, $\rho_1''=\rho_1$ if

$$\frac{\partial F^0}{\partial\rho_1''}=-\frac{C^2}{m'\rho_1''^3}+\frac{M'm'}{\rho_1''^2}=0\ ,\qquad \rho_1''=\rho_1=\frac{C^2}{M'm'^2}\ .$$

Suppose $\rho_1''=\rho_1+\xi$, $R_1''=\eta$; the function F^0 becomes

$$F^0=\frac{\eta^2}{2m'}+\frac{M'm'}{2\rho_1^3}\xi^2+F_3^0\ , \tag{3}$$

where F_3^0 contains only powers of ξ higher than the second. The relations $\xi=\eta=0$ characterize the orbits for which the mutual distances r_0, r_1, r_2 are constant. If $\bar{F}_1=F_1-F^0$, we obtain

$$F=\frac{\eta^2}{2m'}+\frac{M'm'}{2\rho_1^3}\xi^2+\bar{F}_1+F_2+F_3+F_3^0\ . \tag{4}$$

* See (A), §§1, 2; Poincaré, *Les Méthodes Nouvelles*, vol. 1, §19.

In §9, (A), it was seen that if $\bar{F}_1$ is expanded as a power series in the variables u_1, u_2, v_1, v_2, only terms of the second and higher orders occur. It is now a simple matter to write down the equations of variation corresponding to equations (6), §1. These become

$$
\begin{aligned}
&(a)\quad \frac{d\xi}{dt}=\frac{\eta}{m'}, \quad \frac{d\eta}{dt}=-\frac{M'm'}{\rho_1^3}\xi\,;\\
&\frac{d\bar{u}_1}{dt}= \qquad\qquad\qquad A_{12}\bar{v}_1\,,\\
&\frac{d\bar{u}_2}{dt}= B_{12}\bar{u}_1 \qquad\qquad +B_{22}\bar{v}_2\,,\\
&(b)\\
(5)\qquad &\frac{d\bar{v}_1}{dt}=-A_{11}\bar{u}_1 \qquad\qquad -B_{12}\bar{v}_2\,,\\
&\frac{d\bar{v}_2}{dt}= \qquad -B_{11}\bar{u}_2\,;\\
&\frac{d\xi_3}{dt}=\frac{\eta_3}{\bar{m}_1}\,,\\
&\frac{d\xi_6}{dt}=\frac{\eta_6}{\bar{m}_2}\,,\\
&(c)\\
&\frac{d\eta_3}{dt}=-\left[\frac{\lambda^2 m_2 m_0}{\bar{r}_1^3}+\frac{\mu^2 m_1 m_2}{\bar{r}_0^3}+\frac{m_0 m_1}{\rho_1^3}\right]\xi_3+\left[\frac{m_1 m_2 \mu}{\bar{r}_0^3}-\frac{m_2 m_0 \lambda}{\bar{r}_1^3}\right]\xi_6\,,\\
&\frac{d\eta_6}{dt}= \left[\frac{\mu m_1 m_2}{\bar{r}_0^3}-\frac{\lambda m_2 m_0}{\bar{r}_1^3}\right]\xi_3-\left[\frac{m_1 m_2}{\bar{r}_0^3}+\frac{m_2 m_0}{\bar{r}_1^3}\right]\xi_6\,.
\end{aligned}
$$

If equations (5, c) are written in the form

$$
\begin{aligned}
&\frac{d\xi_3}{dt}=\frac{\eta_3}{\bar{m}_1}\,, \quad \frac{d\xi_6}{dt}=\frac{\eta_6}{\bar{m}_2}\,,\\
(6)\qquad &\frac{d\eta_3}{dt}=-C_{11}\xi_3+C_{12}\xi_6\,,\\
&\frac{d\eta_6}{dt}= C_{12}\xi_3-C_{22}\xi_6\,,
\end{aligned}
$$

it is easily seen that the characteristic exponents s must satisfy

$$
(7)\qquad \bar{m}_1\bar{m}_2 s^4+[C_{11}\bar{m}_2+C_{22}\bar{m}_1]s^2+C_{11}C_{22}-C_{12}^2=0\,,
$$

and consequently are all pure imaginaries. It is also evident that the exponents corresponding to (5, *a*) are pure imaginaries; from the calculations of (A), §9, it has been seen that two exponents are pure imaginaries, while the remaining two are real, corresponding to the equations (5, *b*).

In order to apply the generalized theorems of Bohl it only remains to calculate the numbers ϵ_i corresponding to these eight pure imaginary exponents; from (4) it is seen that the terms of the second order in ξ, η, $\bar{x}_3$, $\bar{x}_6$, $\bar{y}_3$, $\bar{y}_6$, occurring in the expansion of F form a positive definite form in these variables, and since the variables employed in (B), §2, are obtained by a real linear transformation, the new corresponding form must also be positive definite (as is easily verified directly). It only remains to calculate the values of ϵ_i corresponding to (5, *b*). The corresponding exponents satisfy

$$s^4+[A_{11}A_{22}+B_{11}B_{22}]s^2+B_{11}A_{22}[A_{11}B_{22}-B_{12}^2]=0\,, \tag{8}$$

and if a solution of (*b*) is assumed in the form

$$\bar{u}_1=\xi_1e^{st}\,,\qquad \bar{u}_2=\xi_2e^{st}\,,\qquad \bar{v}_1=\eta_1e^{st}\,,\qquad \bar{v}_2=\eta_2e^{st}\,,$$

we obtain the relations

$$\begin{aligned}
-s\cdot\xi_1+0\cdot\xi_2+A_{22}\eta_1+0\cdot\eta_2&=0\,,\\
B_{12}\xi_1-s\cdot\xi_2+0\cdot\eta_1+B_{22}\eta_2&=0\,,\\
-A_{11}\xi_1+0\cdot\xi_2-s\cdot\eta_1-B_{12}\eta_2&=0\,,\\
0\cdot\xi_1-B_{11}\xi_2+0\cdot\eta_1-s\cdot\eta_2&=0\,.
\end{aligned} \tag{9}$$

If s satisfies (8), and is of the form $p\sqrt{-1}$, $p>0$, the third line can be omitted; if the values of ξ_1, ξ_2, η_1, η_2 are chosen equal to the corresponding three-rowed determinants remaining,

$$(\xi_1\bar{\eta}_1-\eta_1\bar{\xi}_1)+(\xi_2\bar{\eta}_2-\eta_2\bar{\xi}_2)=-2A_{22}s\{(s^2+B_{11}V_{22})^2+A_{22}B_{11}B_{12}^2\}\,,$$

the dashed constants being the conjugate imaginaries of the corresponding ones with the same subscripts.

Since A_{22} and B_{11} are positive (see (A), §9), it follows that if each of the constants ξ_i, $\bar{\xi}_i$, η_i, $\bar{\eta}_i$ $(i=1, 2)$ is divided by the real constant

$$\sqrt{-2A_{22}s^2[(s^2+B_{11}B_2)^2+A_{22}B_{11}B_{12}^2]}$$

the resulting constants will be of the form required for the application of the theorems of (B), and $\epsilon_i=1/s_i$. Hence the terms of the second order in the expansion of F corresponding to the eight imaginary exponents will be

a positive definite form, and the generalized theorems of Bohl can be applied.* Suppose new variables $x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4, u_1', v_1'$ introduced as in (B), §2, Part I. The function F is analytic in these variables, consequently a constant d can be chosen so small that from (B), II, §4, we have the following theorems:

1. *To every set of values of the region* (R),

$$\sum_{i=1}^{4}(x_i^2+y_i^2)\leqq 2d^2/9\ ,\qquad |v_1'|\leqq d/2\ ,$$

can be adjoined a value $u_1', |u_1'|\leqq d/2$ *such that the set of values* (x_i, y_i, v_1'), u_1' *are the initial values of a trajectory which remains in the region*

$$(C)\qquad \sum_{i=1}^{4}(x_i^2+y_i^2)\leqq d^2\ ,\qquad |u_1'|\leqq d/2\ ,\qquad |v_1'|\leqq d/2$$

for $t>t_0$, *and this value is uniquely determined.*

2. *To every set of values* (x_i, y_i, u_1') *of the region* (R'),

$$\sum_{i=1}^{4}(x_i^2+y_i^2)\leqq 2d^2/9\ ,\qquad |u_1'|\leqq d/2\ ,$$

corresponds a value $v_1', |v_1'|\leqq d/2$, *such that the trajectory with this set of initial values at* $t=t_0$ *lies in the region* (C) *for* $t<t_0$, *and this value is uniquely determined.*

3. *To every set of values of the region*

$$\sum_{i=1}^{4}(x_i^2+y_i^2)\leqq 2d^2/9$$

corresponds a set of values of the region $|u_1'|\leqq d/2, |v_1'|\leqq d/2$, *such that the trajectory with these initial values at* $t=t_0$ *remains in the region* (C) *for all values of* t; *and the values of* u_1', v_1' *are uniquely determined.*

3. Orbits in the neighborhood of the equilateral triangle solutions

From the calculations of (A), §8, it is seen that when the invariant relations characterizing an equilateral triangle solution are satisfied, the function F_1 of (9), §1, becomes

$$F_0=\frac{1}{2\bar{m}}\left[R_1''^2+\frac{C^2}{\rho_1''^2}\right]-\frac{M\bar{m}}{\rho_1''}\ ,$$

(1)

$$\bar{m}=\bar{m}_1+\gamma^2\bar{m}_2=\frac{m_0m_1+m_1m_2+m_2m_0}{M}\ ,\qquad M=m_0+m_1+m_2\ .$$

* (B), Part II, §2.

If ρ is a constant value of ρ_1'' corresponding to a solution in which the mutual distances r_0, r_1, r_2 are constant, then, as in §2,

$$\rho = C^2/M\bar{m}^2 ,$$

and if $\rho_1'' = \rho + \xi$, $R_1'' = \eta$, the function F_0 becomes

$$F_0 = \frac{\eta^2}{2\bar{m}} + \frac{M\bar{m}}{2\rho^3}\xi^2 + \bar{F}_3^0 ,$$

where $\bar{F}_3^0$ contains only powers of ξ greater than the second. If $F_1' = F_1 - F_0$, then

$$F = \frac{\eta^2}{2\bar{m}} + \frac{M\bar{m}}{2\rho^3}\xi^2 + F_1' + F_2 + F_3 + \bar{F}_3^0 .$$

In §10, (A), it was found that if F_1' is expanded as a power series in u_1, u_2, v_1, v_2 only terms of the second and higher orders occur. The equations of variation corresponding to (5, b, §2) have a different form, but the remaining equations of variation are formally similar to the corresponding ones of (5, §2). In §10, (A), it has been seen that the characteristic exponents are either all pure imaginaries, or they contain a group of two pairs of conjugate exponents, according as a certain function of the masses is positive or negative. If the four complex exponents occur, the conditions necessary for the transformation of (B) are seen to be satisfied; it is not necessary to calculate the numbers ϵ_i, since the corresponding terms in F (from $F_0 + F_2$) are known to form a positive definite form; and the linear transformations to the variables used in the theorems of (B) are real. As the function F is analytic within the region considered here, the generalized theorems of Bohl can be applied. Suppose new variables x_1, x_2, x_3, y_1, y_2, y_3, $\bar{u}_1$, $\bar{u}_2$, $\bar{v}_1$, $\bar{v}_2$ introduced as in (B), §2, Part I. The constant d can then be chosen so small that the following theorems hold:

1. *To every set of values of the region* (R_1),

$$\sum_{i=1}^{3} (x_i^2 + y_i^2) \leqq 2d^2/9 , \qquad \bar{v}_1^2 + \bar{v}_2^2 \leqq d^2/4 ,$$

can be adjoined a set of values of the region $\bar{u}_1^2 + \bar{u}_2^2 \leqq d^2/4$, *such that the set of values* $(x_i, y_i, \bar{v}_r)$, $(\bar{u}_r)$ *form the initial values at* $t = t_0$ *of a trajectory which remains in the region*

$$(C') \qquad \sum_{i=1}^{3} (x_i^2 + y_i^2) \leqq d^2 , \qquad \bar{u}_1^2 + \bar{u}_2^2 \leqq d^2/4 , \qquad v_1^2 + v_2^2 \leqq d^2/4$$

for $t > t_0$, *and this set of values is uniquely determined.*

2. *To every set of values* $(x_i, y_i, \bar{u}_r)$ *of the region* (R_1'),

$$\sum_{i=1}^{3} (x_i^2+y_i^2) \leqq 2d^2/9 , \quad \bar{u}_1^2+\bar{u}_2^2 \leqq d^2/4 ,$$

can be adjoined a set of values of the region $\bar{v}_1^2+\bar{v}_2^2 \leqq d^2/4$, *such that the set of values* $(x_i, y_i, \bar{u}_r)$, $(\bar{v}_r)$ *form the initial values at* $t=t_0$ *of a trajectory which remains in* (C') *for* $t<t_0$, *and this set of values is uniquely determined.*

3. *To every set of values of the region*

$$\sum_{i=1}^{3} (x_i^2+y_i^2) \leqq 2d^2/9$$

corresponds a set of values of the region

$$\bar{u}_1^2+\bar{u}_2^2 \leqq d^2/4 , \qquad \bar{v}_1^2+\bar{v}_2^2 \leqq d^2/4$$

such that the trajectory with these initial values at $t=t_0$ *remains in the region* (C') *for all values of* t; *and the set of values* $(\bar{u}_r)$, $(\bar{v}_r)$ *is uniquely determined.*

If we put $\rho=a$ in the expression above for F, it has been found that the characteristic equation corresponding to the analogue of equations (5, b, §2) can be written*

$$s^4+\frac{M}{a^3}s^2+\frac{27}{4a^6}[m_0m_1+m_1m_2+m_2m_0]=0 ,$$

or if $n=\sqrt{M/a^3}$ and $s=an$,

$$M^2a^4+M^2a^2+\frac{27}{4}M\bar{m}=0 ,$$

from which

$$a=\pm\sqrt{-\tfrac{1}{2}\pm\tfrac{1}{2}\sqrt{1-27\bar{m}/M}} .$$

In order for the theorems above to be valid, we must have

$$M-27\bar{m}<0 , \quad \text{or} \quad M^2<27(m_0m_1+m_1m_2+m_2m_0) .$$

The inequality is not satisfied, in particular, when two masses are small in comparison with the remaining one.

It is easily seen that when the last inequality is not satisfied, the roots being all distinct, the characteristic exponents are all pure imaginaries, but the quadratic form obtained in the expansion of F, according to the

* (A), §10.

methods of (B), is indefinite, containing eight positive and two negative terms. Hence the generalized theorems of Bohl cannot be applied here.

4. Conclusion

If the new variables which appear in the theorems of the preceding paragraphs are chosen according to the method of (B), there will be four of the variables corresponding to the pure imaginary exponents which are linear combinations of the variables $\bar{x}_3$, $\bar{x}_6$, $\bar{y}_3$, $\bar{y}_6$; if these new variables are equated to zero, the variables $\bar{x}_3$, $\bar{x}_6$, $\bar{y}_3$, $\bar{y}_6$ will themselves be zero, and the motion will take place in the plane corresponding. Hence the theorems corresponding to those of the two preceding paragraphs, relating to the motion in the plane, can be obtained by supposing four of the variables which correspond to the pure imaginary exponents, for instance x_1, x_2, y_1, y_2, constantly zero.

It follows immediately that the only plane orbits which remain in the neighborhood of the given equilateral triangle orbit ($M<27\bar{m}$) for all values of the time are equilateral triangle orbits; for if the eight equations characterizing an equilateral triangle orbit in the plane $\bar{x}_3=\bar{x}_6=\bar{y}_3=\bar{y}_6=0$ are satisfied, there remain two initial values for the coördinates which can be chosen arbitrarily in a suitably restricted region. But in the plane there are just two initial values, which can be chosen arbitrarily for an orbit remaining in the given neighborhood for all values of t, the orbit being then uniquely determined; since these initial values are linear combinations of the initial values of the original coördinates, the two orbits must coincide.

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EXISTENCE THEOREMS FOR A LINEAR PARTIAL DIFFERENCE EQUATION OF THE INTERMEDIATE TYPE*

BY

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In a recent paper† the author has called attention to three types of linear partial difference equations constructed by analogy with the two types of difference equations in one independent variable which have already been made the object of a considerable amount of study. To the third of these types, which is in the nature of a cross between the linear ordinary difference equation and the linear ordinary q-difference equation, he has ventured to attach the name *linear partial difference equation of the intermediate type.* It is the following special equation of this type to which the reader's consideration is now invited:

$$g(x+1, ry) = p(x, y)g(x, y), \tag{1}$$

where r is a constant, real or complex and different from zero; $p(x, y)$ is a known analytic function of the complex variables x and y; and $g(x, y)$ is the function to be determined. This is the equation of first order of the class (C') spoken of in the paper just now referred to.‡ Some study of the equation of order n of this class has been made by the writer, but the comparative simplicity of the theory of the equation of first order in the face of the rather extreme complexity of that of the more general type of equation, and the fact that somewhat more inclusive theorems can be proved for the former than for the latter, have made it seem desirable to present first a treatment of equation (1).

The method employed below is a natural modification of that introduced by Birkhoff§ in his important paper on the linear ordinary difference equation. In §1 formal series solutions of (1) are found when the coefficient function $p(x, y)$ is a polynomial in x and y. In §2 the existence is established of what may properly be termed principal solutions. Certain extensions of the results of §§1, 2 are indicated in §3.

* Presented to the Society, December 30, 1924; received by the editors in June, 1925.

† *The general theory of a class of linear partial q-difference equations*, these Transactions, vol. 26 (1924), pp. 283-312. This paper will hereinafter be referred to as I.

‡ Loc. cit., p. 284.

§ *General theory of linear difference equations*, these Transactions, vol. 12 (1911), pp. 243-284.

It is appropriate to remark here that results similar to those obtained in the following pages can be proved for the equation in N independent variables

$$g(x+1, r_1y, r_2z, \cdots, r_{N-1}w) = p(x, y, z, \cdots, w)g(x, y, z, \cdots, w).$$

1. The formal series solutions

It is our purpose to show that there exist analytic solutions of the equation (1) when $p(x, y)$ is a given polynomial

$$p(x, y) = p + p_{10}x + p_{01}y + \cdots + p_{lm}x^l y^m.$$

Two symbolic solutions obtained by iterating (1) are the infinite products*

$$p(x-1, y/r)p(x-2, y/r^2)p(x-3, y/r^3)\cdots,$$
$$p^{-1}(x, y)p^{-1}(x+1, ry)p^{-1}(x+2, r^2y)\cdots;$$

certain modifications of these will be the basis of our proof.

Let the degree of $p(x, y)$ in x be μ and its degree in y be ν. It will be assumed that we have

$$p_{\mu 0} \neq 0 \qquad \text{and} \qquad p_{\mu\nu} \neq 0.$$

Under these conditions and if $|r|$ is $\neq 1$, it is found by direct substitution and reckoning of the coefficients that equation (1) has two formal series solutions:†

$$(2)\quad s(x, y) = x^{\mu x}e^{-\mu x}x^{\alpha}y^{\rho}\left[s + \frac{s_{10}}{x} + s_{01}y + \frac{s_{20}}{x^2} + \frac{s_{11}y}{x} + s_{02}y^2 + \cdots\right],$$

$$(3)\quad s'(x, y) = x^{b\cdot} \cdot {}^{-\mu x}r^{\nu(\tau^2-\tau)/2}x^{b}y^{\rho'}\left[s' + \frac{s'_{10}}{x} + \frac{s'_{01}}{y} + \frac{s'_{20}}{x^2} + \frac{s'_{11}}{xy} + \frac{s'_{02}}{y^2} + \cdots\right],$$

where

$$\rho = \frac{\log p_{\mu 0}}{\log r}, \qquad \rho' = \frac{\log p_{\mu\nu}}{\log r}, \quad \text{and} \quad \tau = \frac{\log y}{\log r}.$$

---

* The reciprocal $1/p(x, y)$ will be denoted by $p^{-1}(x, y)$.

† For the actual calculation of the coefficients in the series (2) and (3) use should be made of the expansion

$$\left(1+\frac{1}{x}\right)^{\mu x} = e^{\mu x \log(1+1/x)} = e^{\mu} - \frac{\mu e^{\mu}}{2x} + \frac{\mu(3\mu+8)e^{\mu}}{24x^2} - \cdots.$$

It should be observed that the power of y occurring immediately before the bracketed series in each solution is not a uniquely determined factor; in place of y^{ρ} in (2), for example, we might use σ^x as in the case of the linear ordinary difference equation, or in fact, any function $f(x, y)$ such that $f(x+1, ry) = cf(x, y)$, where c is a constant.

Throughout this paper we shall take the determination of $\log r$ in which the coefficient of $\sqrt{-1}$ is positive or zero and less than 2π.

In general the formal series solutions (2) and (3) fail to converge; it will be shown, however, that solutions of (1) do exist which are analytic (except perhaps for poles) over the entire finite x- and y-planes ($y=0$ excepted for the second solution), and which are asymptotically represented, in a sense presently to be defined, by the formal series (2) and (3).

The most general analytic solution of (1) is given by

$$h(x, y)=g(x, y)a(x, y)\,,$$

where $g(x, y)$ is any particular analytic solution and $a(x, y)$ is any arbitrary analytic function not identically zero and periodic in such a fashion that $a(x+1, ry)=a(x, y)$.

Throughout the remainder of this paper it will be assumed that $|r|$ is greater than unity. If $|r|$ is <1, an entirely analogous treatment is possible. In this second case the rôles which the symbolic solutions $p(x-1, y/r)p(x-2, y/r^2)\cdots$ and $p^{-1}(x, y)p^{-1}(x+1, ry)\cdots$ play in the first case are interchanged. The sort of parallelism that occurs is like that which has been found in the author's earlier paper on partial q-difference equations,* and has been illustrated so well there as to make it unnecessary for us in this paper to give explicit statements of the theorems when $|r|$ is <1.

2. The existence of principal solutions

Let $t(x, y)$ denote the polynomial which $s(x, y)$ becomes upon breaking off the bracketed series in (2) so as to retain only terms whose degree in $1/x$ and y together is less than k, or, more generally, a series obtained from $s(x, y)$ by replacing the bracketed series there by one which has the same terms up to and including those of degree $k-1$, but which converges in the vicinity of the place† $(\infty, 0)$. We may then define a function $q(x, y)$ by the relation

$$t(x+1, ry)=q(x, y)t(x, y);$$

$q(x, y)$ is thus a function of the form $x^{\mu}\,[q+q_{10}/x+q_{01}y+\cdots]$ in which the series in brackets has exactly the same terms of degree less than k in $1/x$ and y as has $x^{-\mu}p(x, y)$. This is seen directly upon comparing

$$q(x, y)=t(x+1, ry)t^{-1}(x, y) \quad \text{and} \quad p(x, y)=s(x+1, ry)s^{-1}(x, y),$$

* I, pp. 291-298.

† We shall think of the complex variables x and y as represented by points in two distinct planes and shall speak of a pair of associated values of x and y as the *place* (x, y).

for $t(x+1, ry)$ is the same as $s(x+1, ry)$, and $t^{-1}(x, y)$ the same as $s^{-1}(x, y)$, up to and including terms of degree $k-1$ in $1/x$ and y. It follows that we have

$$p(x, y)=q(x, y)+m(x, y),$$

where $m(x, y)$ is a series in powers of $1/x$ and y whose lowest degree terms are of degree k or higher, and which converges in the vicinity of $(\infty, 0)$. Hence $n(x, y)$, defined by the equation

$$m(x, y)=q(x, y)n(x, y),$$

is a power series of the same type as $m(x, y)$, and we have

$$p(x, y)=q(x, y)[1+n(x, y)].$$

Before proceeding further with our development it is necessary to make a convenient and not unnatural definition of what we shall mean by asymptotic representation of a function by a series of the type (2). Let $g(x, y)$ be a given function, and consider the difference

$$g(x, y)x^{-\mu z}e^{\mu z}x^{-\alpha}y^{-\rho}-\left[s+\frac{s_{10}}{x}+s_{01}y+\cdots\right],$$

where the bracket contains all the terms in $1/x$ and y up to and including those in $(1/x)^s$ and y^t and only those. *If for each value of s and of t this difference consists of two parts one of which becomes uniformly small of order $x^{-(s+1)}$ as x becomes infinite in a certain region and the other of which becomes uniformly small of order y^{t+1} as y approaches zero in any manner whatsoever, then we say that $g(x, y)$ is asymptotically represented by $s(x, y)$ in that region with respect to x, for $|y|$ small; if the first part becomes uniformly small of order $v^{-(s+1)}$ (where $x=u+\sqrt{-1}v$, u and v real) as v becomes infinite in the region, the representation is said to be asymptotic in that region with respect to v, for $|y|$ small.*

In an entirely similar manner we define asymptotic representation of a function by a series of the type (3) with respect to x or v, except that this time it is *for $|y|$ large.*

The following inequalities are of importance to us:

$$(4)\qquad \left|\sum_{\nu=1}^{\infty}\frac{1}{x-\nu|^s}<\frac{\pi}{2|x|^{s-1}}\right. \qquad (s\geqq 2)$$

for x in the left half of the plane, and

$$\sum_{\nu=1}^{\infty} \frac{1}{|x-\nu|^s} < \frac{2}{|v|^{s-1}}\left[\frac{1}{|v|}+\frac{\pi}{2}\right] \qquad (s \geqq 2) \tag{5}$$

for x in the right half of the plane.*

We are now ready to prove

THEOREM A. *Form the product*

$$p_m(x, y) = p\left(x-1, \frac{y}{r}\right) p\left(x-2, \frac{y}{r^2}\right) \cdots p\left(x-m, \frac{y}{r^m}\right) t\left(x-m, \frac{y}{r^m}\right).$$

As m becomes infinite this product converges, for k sufficiently large, to a definite limit function $g(x, y)$ independent of k. The function $g(x, y)$ is analytic throughout the finite x- and y-planes; its asymptotic form is given by $s(x, y)$ with respect to x in any left half of the x-plane, for $|y|$ small, and with respect to v in any right half of the x-plane, for $|y|$ small; it constitutes a principal solution of the equation (1).

Proof. We may write

$$p_m(x, y) = t(x, y)\bar{p}_m(x, y),$$

where

$$\bar{p}_m(x, y) = \left[t^{-1}(x, y) p\left(x-1, \frac{y}{r}\right) t\left(x-1, \frac{y}{r}\right)\right] \cdot \left[t^{-1}\left(x-1, \frac{y}{r}\right) p\left(x-2, \frac{y}{r^2}\right) t\left(x-2, \frac{y}{r^2}\right)\right] \cdots \left[t^{-1}\left(x-m+1, \frac{y}{r^{m-1}}\right) p\left(x-m, \frac{y}{r^m}\right) t\left(x-m, \frac{y}{r^m}\right)\right].$$

$t(x, y)$ is a polynomial or a series convergent in the vicinity of $(\infty, 0)$ (multiplied by certain factors, cf. (2)). In order to show that $p_m(x, y)$ converges to a limit function it will therefore suffice to prove that $\bar{p}_m(x, y)$ does. To this end our attention will now be directed.

We have expressed $\bar{p}_m(x, y)$ as the product of functions of the type

$$t^{-1}(x+1, ry)p(x, y)t(x, y) = 1+n(x, y),$$

$n(x, y)$ being a power series in $1/x$ and y which converges in the vicinity of $(\infty, 0)$ and has no terms of degree less than k in $1/x$ and y. Let, first of all, k be chosen greater than or equal to 2; then it may be regarded as the sum of two integers l and d, each of which is at least equal to 1; i. e.,

$$k = l+d, \qquad l, d \geqq 1.$$

* The proof of these inequalities is given in some detail by Birkhoff, loc. cit., pp. 248-249.

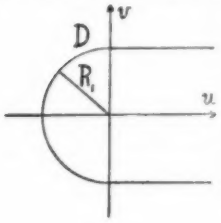
During the proof of convergence k, l, and d are all to be regarded as fixed.

We may write

$$\bar{p}_m(x,y)=\left[1+n\left(x-1,\frac{y}{r}\right)\right]\left[1+n\left(x-2,\frac{y}{r^2}\right)\right]$$

$$\cdots\left[1+n\left(x-m,\frac{y}{r^m}\right)\right]=1+\sum_{k_1=1}^{m}n\left(x-k_1,\frac{y}{r^{k_1}}\right)+\sum_{k_1=1}^{m-1}\sum_{k_2=k_1+1}^{m}n\left(x-k_1,\frac{y}{r^{k_1}}\right)n\left(x-k_2,\frac{y}{r^{k_2}}\right)+\cdots. \tag{6}$$

Now inasmuch as $n(x, y)$ is convergent for (x, y) in the vicinity of $(\infty, 0)$, it follows that if x lies in a certain region D, shown in the accompanying figure and such that $x-i$ $(i=1, 2, 3, \cdots)$ lies outside the circle of radius R_1 about the origin, and y lies inside an associated circle of radius r_2 about the origin in the y-plane, each of the n-series occurring in (6) will be convergent. For the present it will be supposed that x and y lies in these regions of their respective planes.



X-plane

In order to prove the convergence of the multiple series which (6) becomes upon allowing m to increase without limit we shall break each function $n(x-k_i, y/r^{k_i})$ into two parts, the first consisting of all the terms in which y occurs to a power $\geqq l$ and the second made up of the remaining terms. For places (x, y) in the region specified the first part is less in absolute value than $\overline{M}/|r|^{k_1 l}$ and the second is less in absolute value than $\overline{\overline{M}}/|x-k_i|^{d+1}$, where $\overline{M}$ and $\overline{\overline{M}}$ are certain positive constants independent of the value of k_i. The terms of (6), omitting the first, are therefore less in absolute value than the corresponding terms of the following series:

$$\sum_{k_1=1}^{m}\left[\frac{\overline{M}}{|r|^{k_1 l}}+\frac{\overline{\overline{M}}}{|x-k_1|^{d+1}}\right]+\sum_{k_1=1}^{m-1}\sum_{k_2=k_1+1}^{m}\left[\frac{\overline{M}}{|r|^{k_1 l}}+\frac{\overline{\overline{M}}}{|x-k_1|^{d+1}}\right]\cdot\left[\frac{\overline{M}}{|r|^{k_2 l}}+\frac{\overline{\overline{M}}}{|x-k_2|^{d+1}}\right]+\cdots.$$

As m becomes infinite this sum approaches the limit

$$\left[\left(1+\frac{\overline{M}}{|r|^{l}}+\frac{\overline{\overline{M}}}{|x-1|^{d+1}}\right)\left(1+\frac{\overline{M}}{|r|^{2l}}+\frac{\overline{\overline{M}}}{|x-2|^{d+1}}\right)\cdot\left(1+\frac{\overline{M}}{|r|^{3l}}+\frac{\overline{\overline{M}}}{|x-3|^{d+1}}\right)\cdots\right]-1. \tag{7}$$

Since $|r|$ is >1 and both l and d are $\geqq 1$, the bracketed infinite product converges. It follows that $\bar{p}_m(x, y)$, and therefore $p_m(x, y)$, converges absolutely and uniformly to a function analytic in the neighborhood of places (x, y) for which x is in D and $|y|$ is $< r_2$. These limit functions we denote respectively by $\bar{g}(x, y)$ and $g(x, y)$.

Inasmuch as it is possible to write

$$p_m(x, y) = p\left(x-1, \frac{y}{r}\right) p\left(x-2, \frac{y}{r^2}\right) \cdots p\left(x-h, \frac{y}{r^h}\right) p_{m-h}\left(x-h, \frac{y}{r^h}\right)$$

and to give the same argument for $p_{m-h}(x-h, y/r^h)$ as has been given above for $p_m(x, y)$, h being arbitrarily large, we conclude that $p_m(x, y)$ converges also for places (x, y) for which x is finite but not in D or $|y|$ is finite but $>r_2$, or both these conditions obtain. This completes the proof of convergence.

The function $g(x, y)$ is independent of k, because if k' is any second value for k and $t'(x, y)$ the function corresponding to $t(x, y)$, we may write

$$\begin{aligned} p'_m(x, y) = t(x, y) &\left[t^{-1}(x, y) p\left(x-1, \frac{y}{r}\right) t\left(x-1, \frac{y}{r}\right)\right] \\ &\cdot \left[t^{-1}\left(x-1, \frac{y}{r}\right) p\left(x-2, \frac{y}{r^2}\right) t\left(x-2, \frac{y}{r^2}\right)\right] \\ \cdots &\left[t^{-1}\left(x-m+1, \frac{y}{r^{m-1}}\right) p\left(x-m, \frac{y}{r^m}\right) t'\left(x-m, \frac{y}{r^m}\right)\right]. \end{aligned}$$

Then if $p'_m(x, y)$ be expanded in a sum like (6) and m be allowed to become infinite, the resulting multiple series for $g(x, y)$ will be term by term identical with that obtained above.

We have still to examine the asymptotic form of $g(x, y)$. First of all let it be observed that the expression (7) is less than

$$\left| e^{\sum_{\nu=1}^{\infty} \frac{\overline{M}}{|r|^{\nu l}} + \sum_{\nu=1}^{\infty} \frac{\overline{\overline{M}}}{|x-\nu|^{d+1}}} - 1 \right|.$$

Hence the limit function $\bar{g}(x, y)$ differs from 1 by a quantity less in absolute value than

$$\left| e^{\frac{\overline{M}}{|r|^l - 1} + \sum_{\nu=1}^{\infty} \frac{\overline{\overline{M}}}{|x-\nu|^{d+1}}} - 1 \right|.$$

By taking k, and with it l, sufficiently large, however, we can make the first term in the exponent of e as small as we please. Yet it has been shown that $\bar{g}(x, y)$ is independent of k. It therefore follows that we have

$$|\bar{g}(x,y)-1| \leqq \left| e^{\sum_{\nu=1}^{\infty} \frac{\overline{M}}{|x-\nu|^{d+1}}} - 1 \right|.$$

By virtue of (4) if x lies in any left half-plane, the expression inclosed in bars on the right has the form $M(x)/x^d$ where $M(x)$ is a bounded function. Thus for x in D and in any left half-plane, and for y inside the circle of radius r_2 about the origin, we have

$$\bar{g}(x,y)=1+\frac{M(x,y)}{x^d}, \qquad |M(x,y)|<M,$$

M being a certain positive number. Similarly, for x in D and in any right half-plane and for $|y|<r_2$, we have by (5)

$$\bar{g}(x,y)=1+\frac{M'(x,y)}{y^d}, \qquad |M'(x,y)|<M',$$

where M' is a positive constant. Using the first of these expressions for $\bar{g}(x, y)$ we find

$$g(x,y)=t(x,y)\bar{g}(x,y)=t(x,y)+t(x,y)\frac{M(x,y)}{x^d}$$

$$=x^{\mu z}e^{-\mu z}x^{\sigma}y^{\rho}\left\{\left(s+\frac{s_{10}}{x}+s_{01}y+\cdots\right)+\frac{M(x,y)}{x^d}\left(s+\frac{s_{10}}{x}+s_{01}y+\cdots\right)\right\}.$$

Now if the series (or polynomial) contained in the first parentheses within the braces be broken off so as to retain only those terms in which the exponent of $1/x$ is s or less and that of y is t or less, the remainder may be regarded as consisting of two pieces, one of which is infinitesimal in x to the order $s+1$ and the other infinitesimal in y to the order $t+1$. At the same time the second term within the braces can be made infinitesimal in x to the order $s+1$ by taking k, and with it d, sufficiently large. The reasoning and result are similar if we use the second of the above expressions for $\bar{g}(x, y)$. Hence $g(x, y)$ is represented asymptotically by $s(x, y)$ with

respect to x in any left half-plane and with respect to v in any right half-plane, for $|y|$ small.

It remains for us to justify the term *principal solution* which has been applied to the function $g(x, y)$. To do this it will suffice to show that the properties which this solution possesses characterize it completely. Let it be supposed, then, that there exists a second solution $g'(x, y)$ of (1) which possesses the same properties as $g(x, y)$. The quotient $g(x, y)/g'(x, y) = a(x,y)$ is clearly an analytic function except perhaps for poles over the entire finite x- and y-planes; furthermore it is periodic in such a way that $a(x+1, ry) = a(x, y)$. Consider any place (x_0, y_0) for which x_0 and y_0 are finite; the value of the quotient $a(x, y)$ at this place is the same as its value at $(x_0-1, y_0/r)$, and at $(x_0-2, y_0/r^2)$, etc. But inasmuch as $g(x, y)$ and $g'(x, y)$ both have the asymptotic form (2) for x in any left half-plane and $|y|$ small, the quotient $a(x, y)$ approaches the value 1 as (x, y) traverses the sequence (x_0, y_0), $(x_0-1, y_0/r)$, $(x_0-2, y_0/r^2)$, $\cdots$. It follows that $a(x_0, y_0)$ is itself 1, and in consequence we have $g(x, y) \equiv g'(x, y)$.

Using the methods which have been employed above we can establish a second theorem. For this purpose it is necessary that we define a function $t'(x, y)$ which bears the same relation to $s'(x, y)$ that $t(x, y)$ does to $s(x, y)$; i. e., $t'(x, y)$ is a series convergent in the vicinity of (∞, ∞) and having the same terms as $s'(x, y)$ up to and including those of degree $k-1$ in $1/x$ and $1/y$ together. The statement of the theorem follows.

THEOREM B. *Form the product*

$$p'_m(x, y) = p^{-1}(x, y)p^{-1}(x+1, ry) \cdots p^{-1}(x+m-1, r^{m-1}y)t'(x+m, r^m y).$$

As m becomes infinite this product converges, for k sufficiently large, to a definite limit function $h(x, y)$, independent of k. The function $h(x, y)$ is analytic throughout the finite x-plane and the extended y-plane with the exception of $y=0$, and except at places which are poles of $p^{-1}(x, y)$, or of $p^{-1}(x+1, ry)$, or $\cdots$; at these places the function $h(x, y)$ itself has poles. The asymptotic form of $h(x, y)$ is given by $s'(x, y)$ with respect to x in any right half of the x-plane, for $|y|$ large, and with respect to v in any left half of the x-plane, for $|y|$ large. The function $h(x, y)$ constitutes a second principal solution of the equation (1).

The relation between the solutions. If we define the function $a(x, y)$ by means of the equation

$$g(x, y) = h(x, y)a(x, y),$$

it is clear that $a(x, y)$ is periodic in such a fashion that $a(x+1, ry) = a(x, y)$. It is also evident that $a(x, y)$ is analytic except possibly for poles over the

region consisting of the entire finite x-plane and of the y-plane between circles r_2 and R_2 about the origin, where r_2 is arbitrarily small and R_2 arbitrarily large. That no such poles can actually occur, however, is easily demonstrated by methods already employed elsewhere* by the writer. A further analysis of the periodic function $a(x, y)$ seems feasible only in a very restricted case, and we shall not attempt it here.

3. A MORE GENERAL PROBLEM

The seeming lack of symmetry in the formal series solutions (2) and (3) is only apparent, for if we had begun by supposing that in the polynomial $p(x, y)$ the lowest power of y to occur is y^λ and that $p_{\mu\lambda}$ is different from zero, we should have had in place of (2) the formal solution

$$s(x, y) = x^{\mu_2} e^{-\mu_2 r^{\lambda(\nu_2-\nu)/2} x^{\nu_2}} y^{\rho} \left[s + \frac{s_{10}}{x} + s_{01} y + \cdots \right].$$

After this remark it becomes clear that $p(x, y)$ may just as well be taken to be a polynomial in x, $1/x$, y, and $1/y$, it being essential only that $p(x, y)$ be expressible in the forms

$x^\mu y^\lambda$ [polynomial in $1/x$ and y with constant term $\neq 0$],

$x^\mu y^\nu$ [polynomial in $1/x$ and $1/y$ with constant term $\neq 0$].

Furthermore it may be observed that formal series solutions can be found and existence theorems analogous to those of §1 proved as above if the coefficient function in (1) is any function whatever expressible in each of the following forms, the first valid in the vicinity of $(\infty, 0)$ and the second valid in the vicinity of (∞, ∞):

$x^\mu y^\lambda$ [convergent power series in $1/x$ and y with constant term $\neq 0$],

$x^\mu y^\nu$ [convergent power series in $1/x$ and $1/y$ with constant term $\neq 0$].

The only difference in the results is that the solutions obtained may be meromorphic only in limited regions about $(\infty, 0)$ and (∞, ∞) respectively.

* Cf. I, pp. 304-305.

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AN ALGEBRA OF SEQUENCES OF FUNCTIONS, WITH AN APPLICATION TO THE BERNOULLIAN FUNCTIONS*

BY

E. T. BELL

On account of its simplicity, its immediate applicability to many functions already in existence, and its suggestiveness with regard to generalizations of known functions or the creation of new, the algebra in question merits the somewhat detailed exposition in Part I. As its use is perhaps best seen from an example, we have sketched briefly in Part II the outlines of a new theory of the relations between the Bernoullian and Eulerian functions of any arguments and any ranks, showing that by means of the algebra all necessary computations are reduced to a minimum. These functions are of two variables, of which one is complex and the other, the rank, a positive integer. The algebra establishes a simple isomorphism between the theory of relations between the functions and the like for the ordinary sine and cosine. (It is necessary here to distinguish between ordinary and umbral circular functions; the nature of the distinction appears presently.) In a paper which I expect to publish later there is a more detailed application of this algebra to certain new functions, suggested by the algebra, of three variables, of which two are complex and the third a positive integer. For unit values of one of the complex variables these functions degenerate to the Bernoullian and Eulerian functions of Part II; for unit values of the other, they become certain polynomials, of considerable importance in the arithmetical theory of quadratic forms, discussed on several occasions by Hermite, Weierstrass and, more recently, by Bulyguin and Gruder. In this application there are simple isomorphisms with the circular, the hyperbolic and the elliptic functions. In a third paper, to appear shortly, I have shown how the algebra gives at once the complete theory of the relations between the functions of Spitzer, which include as special cases the Bessel *coefficients*. Here there is simple isomorphism with the exponential function. By a generalization of the exponential function, $\sum_m x^{m+\nu}/\Gamma(m+\nu)$, in which x, ν are complex, I have shown in a paper not yet published that the algebra is readily applicable to the Bessel *functions*. The algebra therefore is of considerable utility. Its chief use, how-

* Presented to the Society, September 11, 1925; received by the editors in April, 1925.

ever, is its suggestiveness as to new functions to be investigated, and its immediate forecasts of the results to be expected. For example, in a paper to be published elsewhere, I have shown that the ultra-Bernoullian functions of Krause and Appell (functions of two complex variables) find their ready generalizations and complete theory by a simple inspection of the symbolic formulas in Part I, and that the number of distinct generalizations possible is infinite. Of rather more importance is the similar generalization of the functions of Olivier, also suggested by the algebra, as these generalizations must appear sooner or later in the study of any sequences of functions (cf. §13).

The umbral calculus of Blissard is used freely in the sequel. Brief but sufficient synopses of the main processes of this powerful but neglected instrument being readily available in the places cited,* we shall assume them known. We presuppose also, §7, end, the legitimacy of operating with infinite series in the manner established in a former paper,† but this assumption affects nothing in Part II or in any previous results in Part I. If preferred all reference to infinite series may be obviated, and operations with series may be replaced, as recently shown by Wedderburn,‡ by others in a certain algebra with an infinite *basis*. Or again, the theorems, when found by either of these methods, can be proved independently if wished by an easy application of mathematical induction.

Part II is not intended in any way as a systematic treatment of the functions discussed, but merely as a collection of examples illustrating some of the results of Part I. Anyone skimming the paper is advised to ascertain from the context what letters denote umbrae; otherwise the point of most of the formulas will be missed.

Part I. The symmetric functions φ, ψ, χ

1. A sequence of functions implies in the definition of a particular general element of the sequence the presence of at least one positive or negative integer, and conversely if in the definition of the general element of a set of functions there occurs at least one integer, the functions of the set can be ordered with respect to this integer into a sequence. If the integer is negative a suitable change in the definition or notation of the functions replaces it by a positive integer. The positive integer is called

* (1) *Umbral symmetric functions* . . . , Mathematische Zeitschrift, vol. 19 (1923), pp. 35-6, §§1, 2; (2) *A revision of the Bernoullian functions*, Bulletin of the American Mathematical Society, vol. 28 (1922), pp. 443-44, §1.

† *Euler algebra*, these Transactions, vol. 25 (1923), pp. 135-54.

‡ These Transactions, vol. 26 (1924), p. 421, formula (9.21).

the rank. Hence the ranks in any sequence of functions can be taken as $0, 1, 2, \cdots$.

With respect to a fixed positive integer m as modulus the ranks of the functions can be segregated into m classes. The functions of the rth class, $0 \leqq r < m$, are those whose ranks are $km+r$ $(k=0, 1, 2, \cdots)$.

The algebra which we are about to construct is fixed by the integer m, and for a definite m is said to be of order m. This is not the usual definition of the order of an algebra, but is closely analogous to it. Instead of m basal *units* of an algebra we have here $m+1$ *types* of symmetric functions, as will appear presently. When $m=2$, the case with which we shall be concerned here, the algebra is simply isomorphic to the theory of either the circular or the hyperbolic functions; when $m=1$ the isomorphism is with the function $\exp ax$, $a \equiv$ an umbra. Now it is well known that the entire theory of the circular (or hyperbolic) functions is implied by (1) the theorem that x^2+y^2, where x is real and y real (or pure imaginary), reproduces itself with respect to multiplication, and (2) the integer solutions of $x^2+y^2=1$. A corresponding pair of theorems unifies the algebras for all orders $m \geqq 2$: (1) the norm of any algebraic integer is self-reproductive under multiplication, (2) the units in the associated field of degree m, when for this (algebraic number) field is taken that defined by a primitive mth root of either 1 or -1. The corresponding functions for $m>2$ are in the simplest instance those of Olivier, in the general case the umbral generalization of these which is indicated in §13. This is not further developed here, and is mentioned merely to show the simple, common basis underlying the algebras of all orders >1. The connection with sequences of the following sections is made in §8.

2. Let each of e_j, e_j' $(j=1, 2, \cdots)$ denote a definite one of $1, -1$. Let first u_j, v_j $(j=1, 2, \cdots)$ be ordinaries (elements of any field), and define φ, ψ, χ by

$$\varphi_{2n}(u) = u^{2n}\ , \quad \psi_{2n+1}(v) = v^{2n+1}\ ,$$

$$(1)\quad \varphi_n(u_1, u_2, \cdots, u_r) \equiv \Sigma(u_1+e_2u_2+e_3u_3+\cdots+e_ru_r)^n\ ,$$

$$(2)\quad \psi_n(v_1, v_2, \cdots, v_s) \equiv \Sigma e_2'e_3'\cdots e_s'(v_1+e_2'v_2+e_3'v_3+\cdots+e_s'v_s)^n\ ,$$

$$(3)\quad \chi_n(u_1, u_2, \cdots, u_r; v_1, v_2, \cdots, v_s) \equiv \Sigma e_1'e_2'\cdots e_s'(u_1+e_2u_2+ \cdots+e_ru_r+e_1'v_1+e_2'v_2+\cdots+e_s'v_s)^n\ ,$$

where $n \geqq 0$ is an integer, and the summations refer to all possible values of the e_j, e_k'. To simplify the writing, put henceforth

$$u_1, u_2, \cdots, u_r \equiv U_r \equiv U\ , \quad v_1, v_2, \cdots, v_s \equiv V_s \equiv V$$

and similarly for accented letters, $u_1', u_2', \cdots, u_p' \equiv U_p'$, etc., the U, V without suffixes being used when it is unnecessary to distinguish between even and odd values of r, s. Each of $u, u', u'', \cdots, v, v', v'', \cdots$ (without a suffix) denotes a *single* ordinary or a *single* umbra, according to the context in which it occurs. The functions (1), (2), (3) differ from those denoted in previous papers by the same letters. For functions of precisely two arguments we have the useful formulas

$$\chi_n(u; v) = \psi_n(u, v), \quad \chi_{2n+1}(u; v) = \psi_{2n+1}(v, u) . \tag{4}$$

From (1)–(3) we write down the following generators, in which the sines and cosines on the right are umbral:

$$2^{r-1} \prod_{j=1}^{r} \cos u_j x = \cos \varphi(U_r)x , \tag{5}$$

$$2^{2s-1}(-1)^s \prod_{j=1}^{2s} \sin v_j x = \cos \psi(V_{2s})x , \tag{6}$$

$$2^{2s}(-1)^s \sum_{j=1}^{2s+1} \sin v_j x = \sin \psi(V_{2s+1})x , \tag{7}$$

$$2^{r+2s-1}(-1)^s \prod_{j=1}^{r} \cos u_j x \cdot \prod_{k=1}^{2s} \sin v_k x = \cos \chi(U_r ; V_{2s})x , \tag{8}$$

$$2^{r+2s}(-1)^s \prod_{j=1}^{r} \cos u_j x \cdot \prod_{k=1}^{2s+1} \sin v_k x = \sin \chi(U_r ; V_{2s+1})x . \tag{9}$$

It follows that $\varphi_{2n}(U_r)$ is symmetric in $u_1, u_2, \cdots, u_r$; $\psi_n(V_s)$ is symmetric in $v_1, v_2, \cdots, v_s$; $\chi_n(U_r; V_s)$ is symmetric in $u_1, u_2, \cdots, u_r$ and in $v_1, v_2, \cdots, v_s$. Notice that in (5)–(9) only those $\psi_n(V_s), \chi_n(U_r; V_s)$ occur in which n, s are of like parity, also that only φ's of even rank appear. This holds in all subsequent formulas.

Next let the u_j, v_k be umbrae. Then by the elements of the umbral calculus as developed in the papers cited, it follows that (5)-(9) are valid for the new interpretations of (1)-(3). This remark is necessary for the reading of the formulas in the next section and thereafter. We recall that any umbral formula implies as a special case the same formula when all letters therein are interpreted to be ordinaries.

3. The φ, ψ, χ have three kinds of addition theorems, the first of which expresses a function of $m+n$ arguments $(m, n > 0)$ as a bilinear form in functions of m' and n' arguments, where $m'+n' = m+n$. These theorems are found by replacing in (5)-(9) the u_j, v_k by new arguments u_j', v_k', and (r, s) by (p, t) $(r, s, p, t > 0)$, multiplying member by member the

identities thus obtained by the old, reducing the products by inspection of (5)-(9) to the same forms as these, and using (4) when necessary to replace on the right χ by ψ. Comparing coefficients of like powers of x in the results we get in this way the following set of addition theorems of the first kind:

$$\varphi_{2n}(U_r, U_p') = \varphi_{2n}(\varphi(U_r), \varphi(U_p')) , \tag{10}$$

$$\chi_{2n}(U_r, U_p'; V_{2s}, V_{2t}') = \varphi_{2n}(\chi(U_r; V_{2s}), \chi(U_p'; V_{2t}')) , \tag{11}$$

$$\chi_{2n}(U_r, U_p'; V_{2s+1}, V_{2t+1}') = \psi_{2n}(\chi(U_r; V_{2s+1}), \chi(U_p'; V_{2t+1}')) , \tag{12}$$

$$\chi_{2n}(U_r, U_p'; V_{2s}) = \varphi_{2n}(\varphi(U_r), \chi(U_p'; V_{2s})) , \tag{13}$$

$$\chi_{2n+1}(U_r, U_p'; V_{2s+1}) = \psi_{2n+1}(\varphi(U_r), \chi(U_p'; V_{2s+1})) , \tag{14}$$

$$\chi_{2n+1}(U_r, U_p'; V_{2s}, V_{2t+1}') = \psi_{2n+1}(\chi(U_r; V_{2s}), \chi(U_p'; V_{2t+1})) . \tag{15}$$

It is convenient to have explicitly the special forms obtained upon suppressing one or both of U_r, U_p' and applying

$$\chi_{2n}(0; V_{2s}) = \psi_{2n}(V_{2s}) , \quad \chi_{2n+1}(0; V_{2s+1}) = \psi_{2n+1}(V_{2s+1}) \tag{16}$$

to the results. We find

$$\psi_{2n}(V_{2s}, V_{2t}') = \varphi_{2n}(\varphi(V_{2s}), \psi(V_{2t}')) ,$$

$$\psi_{2n}(V_{2s+1}, V_{2t+1}') = \psi_{2n}(\psi(V_{2s+1}), \psi(V_{2t+1}')) ,$$

$$\psi_{2n+1}(V_{2s}, V_{2t+1}') = \psi_{2n+1}(\psi(V_{2s}), \psi(V_{2t+1}')) ;$$

and for the χ functions of even rank,

$$\chi_{2n}(U_r; V_{2s}) = \varphi_{2n}(\varphi(U_r), \psi(V_{2s})) ,$$

$$\chi_{2n}(U_r; V_{2s}, V_{2t}') = \varphi_{2n}(\psi(V_{2s}), \chi(U_r; V_{2t}')) ,$$

$$\chi_{2n}(U_r; V_{2s+1}, V_{2t+1}') = \psi_{2n}(\psi(V_{2s+1}), \chi(U_r; V_{2t+1}')) ;$$

while for the odd ranks,

$$\chi_{2n+1}(U_r; V_{2s+1}) = \psi_{2n+1}(\varphi(U_r), \psi(V_{2s+1})) ,$$

$$\chi_{2n+1}(U_r; V_{2s}, V_{2t+1}') = \psi_{2n+1}(\psi(V_{2s}), \chi(U_r; V_{2t+1}')) ,$$

$$\chi_{2n+1}(U_r; V_{2s+1}, V_{2t}') = \psi_{2n+1}(\psi(V_{2s+1}), \chi(U_r; V_{2t}')) .$$

Suppression of U_r or U_p' is equivalent to replacing each u_j or each u_j' by zero. We cannot similarly delete either V_s or V_t', for this would be equivalent to taking $v_k=0$ or $v_k'=0$ $(k=1, 2, \cdots)$ in (6)-(9), which obviously is not permissible.

4. The second type of addition theorem expresses the functions of arguments u_j+u_j' linearly in terms of functions of arguments u_j and u_j' $(j=1, 2, \cdots)$ respectively. They are found by replacing any u_j or v_k in (5)-(9) by the sum of two others, expanding the left of the identities so obtained by the addition theorems for the sine and cosine, and using (5)-(9) to reduce the results to umbral sines or cosines of new functions φ, ψ, χ. We find (with $U\equiv U_r$, $V\equiv V_s$ as explained in §2),

$$2\varphi_{2n}(U, u'+u'')=\varphi_{2n}(U, u', u'')-\chi_{2n}(U; u', u'') , \tag{17}$$

$$2\psi_n(V, v'+v'')=\chi_n(v'; V, v'')+\chi_n(v''; V, v') , \tag{18}$$

$$2\chi_n(U, u'+u''; V)=\chi_n(U, u', u''; V)-\chi_n(U; V, u', u'') , \tag{19}$$

$$2\chi_n(U; V, v'+v'')=\chi_n(U, v'; V, v'')+\chi_n(U, v''; V, v') . \tag{20}$$

Applying either of (19), (20) to the other we get

$$\begin{aligned} &4\chi_n(U, u'+u''; V, v'+v'') \\ &=\chi_n(U, u', u'', v'; V, v'')+\chi_n(U, u', u'', v''; V, v') \\ &\quad-\chi_n(U, v'; V, u', u'', v'')-\chi_n(U, v''; V, u', u'', v') . \end{aligned} \tag{21}$$

5. There are two types of subtraction theorems. The second is indicated in §6; the first is found similarly to the preceding addition theorems, or directly from them by means of (29), which is obvious independently. We have

$$2\varphi_{2n}(U, u'-u'')=\varphi_{2n}(U, u', u'')+\chi_{2n}(U; u', u'') , \tag{22}$$

$$2\psi_n(V, v'-v'')=-\chi_n(v'; V, v'')+\chi_n(v''; V, v') , \tag{23}$$

$$2\chi_n(U, u'-u''; V)=\chi_n(U, u', u''; V)+\chi_n(U; V, u', u'') , \tag{24}$$

$$2\chi_n(U; V, v'-v'')=-\chi_n(U, v'; V, v'')+\chi_n(U, v''; V, v') , \tag{25}$$

and hence, from the last two,

$$\begin{aligned} &4\chi_n(U, u'-u''; V, v'-v'') \\ &\quad=-\chi_n(U, u', u'', v'; V, v'')+\chi_n(U, u' u'', v''; V, v') \\ &\quad\quad-\chi_n(U, v'; V, u', u'', v'')+\chi_n(U, v''; V, u', u'', v') . \end{aligned} \tag{26}$$

Similarly from (19), (25) we get

$$(27)\quad 4\chi_n(U, u'+u''; V, v'-v'') = -\chi_n(U, u', u'', v'; V, v'')+\chi_n(U, u', u'', v''; V, v') + \chi_n(U, v'; V, u', u'', v'')-\chi_n(U, v''; V, u', u'', v') ,$$

and from (20), (24),

$$(28)\quad 4\chi_n(U, u'-u''; V, v'+v'') = \chi_n(U, u', u'', v'; V, v'')+\chi_n(U, u', u'', v''; V, v') + \chi_n(U, v'; V, u', u'', v'')+\chi_n(U, v''; V, u', u'', v') .$$

Combine (17)-(25) and find

$$(29)\quad \psi_n(V, -v)=-\psi_n(V, v),\ \chi_n(U; V, -v)=-\chi_n(U; V, v) ,$$

$$(30)\quad \varphi_{2n}(U, u', u'')=\varphi_{2n}(U, u'+u'')+\varphi_{2n}(U, u'-u'') ,$$

$$(31)\quad \chi_{2n}(U; v', v'')=\varphi_{2n}(U, v'-v'')-\varphi_{2n}(U, v'+v'') ,$$

$$(32)\quad \chi_n(u; V, v)=\psi_n(V, u+v)-\psi_n(V, u-v) ,$$

$$(33)\quad \chi_n(U, u', u''; V)=\chi_n(U, u'+u''; V)+\chi_n(U, u'-u''; V) ,$$

$$(34)\quad \chi_n(U; V, v', v'')=\chi_n(U, v'-v''; V)-\chi_n(U, v'+v''; V) ,$$

$$(35)\quad \chi_n(U, u; V, v)=\chi_n(U; V, u+v)-\chi_n(U; V, u-v) ,$$

and hence from (29), (32), (35)

$$(36)\quad \chi_n(u; V, v)=\psi_n(V, v+u)+\psi_n(V, v-u) ,$$

$$(37)\quad \chi_n(U, u; V, v)=\chi_n(U; V, v+u)+\chi_n(U; V, v-u) .$$

To condense the similar formulas obtained from (21), (26)-(28), denote the arguments of the χ_n occurring in the left of these identities by A, B, C, D respectively, and those in the right of (26), in the order in which they are written, by a, b, c, d respectively. Then

$$(38)\quad \chi_n(a)=\chi_n(A)-\chi_n(B)-\chi_n(C)+\chi_n(D) ,$$

$$(39)\quad \chi_n(b)=\chi_n(A)+\chi_n(B)+\chi_n(C)+\chi_n(D) ,$$

$$(40)\quad \chi_n(c)=-\chi_n(A)-\chi_n(B)+\chi_n(C)+\chi_n(D) ,$$

$$(41)\quad \chi_n(d)=-\chi_n(A)+\chi_n(B)-\chi_n(C)+\chi_n(D) ,$$

giving the interesting formula

$$\chi_n^2(a)+\chi_n^2(b)+\chi_n^2(c)+\chi_n^2(d)=4[\chi_n^2(A)+\chi_n^2(B)+\chi_n^2(C)+\chi_n^2(D)]\ , \tag{42}$$

which recalls the identity on which Jacobi founded the theory of elliptic functions. Some curious applications of this identity will be given in other papers; applied to the Bernoullian and Eulerian functions of Part II it yields a remarkable system of relations between squares of sums of these functions which will be developed in a separate note.

6. The addition theorems of the third kind are the resultants of those of the first two kinds, and are among the most important processes of this algebra in its applications. They are so numerous, however, that we shall not take space to write out the set, which can be found from §§3-5, but will merely indicate their nature and write down the principal transformations for obtaining them. Any systematic investigation of a set of sequences of functions presupposes the complete system of such theorems. To derive them we transform the right hand members of (17)-(37) by means of all the formulas in §3, recalling from §2 that $\varphi_{2n}(u)=u^{2n}$, $\psi_{2n+1}(v)=v^{2n+1}$. With the exception of those required for (21), (26)-(28), (38)-(42), the complete set of necessary transformation formulas is

$$\varphi_{2n}(U, u', u'')=\varphi_{2n}(\varphi(U), \varphi(U', u''))=\varphi_{2n}(u', \varphi(U, u''))\ ,$$

$$\chi_{2n}(U; v', v'')=\varphi_{2n}(\varphi(U), \psi(v', v''))=\psi_{2n}(v', \chi(U, v''))\ ,$$

$$\chi_n(u; V, v)=\psi_n(\psi(u, v), \psi(V))=\psi_n(v, \chi(u; V))\ ,$$

$$\chi_{2n}(u; V_{2s+1}, v)=\varphi_{2n}(u, \psi(V_{2s+1}, v))\ ,$$

$$\chi_{2n+1}(u; V_{2s}, v)=\psi_{2n+1}(u, \psi(V_{2s}, v))\ .$$

For the excepted formulas there is a similar but longer list written down in the same way. An example of an addition theorem of the third kind is, by (17) and the above,

$$2\varphi_{2n}(U, u'+u'')=\varphi_{2n}(u', \varphi(U, u''))-\psi_{2n}(u', \chi(U; u''))\ ;$$

which by (4) has the special case

$$2\varphi_{2n}(u, u'+u'')=\varphi_{2n}(u, \varphi(u, u''))-\psi_{2n}(u', \psi(u, u''))\ .$$

In the theory of Bernoullian functions and of many others the special cases of all the addition theorems in which each of $U\equiv U_r$, $V\equiv V_s$ contains not more than two umbrae are fundamental.

Clearly all of the results in §§ 2-6 can be verified without reference to (4)-(9). To do so, however, merely masks their straightforward origin and destroys much of their pliability in use.

7. The processes of generalization outlined in this section unite, in the theory of any given set of sequences, numerous theorems into a single statement and at the same time give a wide extension to isolated results. By means of the umbral calculus any identity or equality between ordinaries can be immediately interpreted as the like between umbrae. The simple rules for performing and interpreting the umbral processes of a field, including umbral differentiation, were summarized in the papers cited in the introduction. We proceed to write out a few of the most frequently used umbral formulas obtainable in this way.

Let first $f(x)$ be a polynomial in the ordinary x, and λ, μ umbrae, a, b ordinaries independent of x, and recall the definition of the *rth umbral derivative of* λ_n *with respect to* λ:

$$D_\lambda^r \lambda^n \equiv D_\lambda^r \lambda_n \equiv \frac{n!}{(n-r)!}\lambda^{n-r} \equiv \frac{n!}{(n-r)!}\lambda_{n-r} .$$

This coincides with the *ordinary derivative* when and only when λ_n is its special case, the nth power of the ordinary λ. It is in general *not* a limiting process. We shall denote the rth ordinary derivative of f with respect to x by the superscript (r), thus $f^{(r)}$, using f', f'' when $r=1, 2$. The sign of implication is as usual .⊃. .

Then evidently for r, s constant positive integers and $n=0, 1, 2, \cdots$ we have the following implications:

$$(43) \qquad a\lambda^{2ns} = b\mu^{2ns} \;.\supset.\; a[f(x+\lambda^s)+f(x-\lambda^s)] = b[f(x+\mu^s)+f(x-\mu^s)] ,$$

$$(44) \qquad a\lambda^{(2n+1)s} = b\mu^{(2n+1)s} \;.\supset.\; a[f(x+\lambda^s)-f(x-\lambda^s)] = b[f(x+\mu^s)-f(x-\mu^s)] ,$$

$$(45) \qquad a\lambda^{ns} = b\mu^{ns} \;.\supset.\; af(x+\lambda^s) = bf(x+\mu^s) ,$$

$$(46) \qquad a\lambda^n = bD_\mu^r \mu^n \;.\supset.\; af(x+\lambda) = bf^{(r)}(x+\mu) .$$

The next also are easily seen, and each is of frequent occurrence:

$$(47) \qquad ax^{2r}\sin\lambda x = b\sin\mu x \;.\supset.\; b\mu^{2n+1} = a(-1)^r D_\lambda^{2r}\lambda^{2n+1} ,$$
$$.\supset.\; a(-1)^r[f^{(2r)}(x+\lambda)-f^{(2r)}(x-\lambda)] = b[f(x+\mu)-f(x-\mu)] ;$$

$$(48) \qquad ax^{2r+1}\sin\lambda x = b\cos\mu x \;.\supset.\; -b\mu^{2n} = a(-1)^r D_\lambda^{2r+1}\lambda^{2n} ,$$
$$.\supset.\; a(-1)^r[f^{(2r+1)}(x+\lambda)-f^{(2r+1)}(x-\lambda)] = -b[f(x+\mu)+f(x-\mu)] ;$$

$$(49) \qquad ax^{2r}\cos\lambda x = b\cos\mu x \;.\supset.\; b\mu^{2n} = a(-1)^r D_\lambda^{2r}\lambda^{2n} ,$$
$$.\supset.\; a(-1)^r[f^{(2r)}(x+\lambda)+f^{(2r)}(x-\lambda)] = b[f(x+\mu)+f(x-\mu)] ;$$

(50) $ax^{2r+1}\cos\lambda x = b\sin\mu x \,.\supset.\, b\mu^{2n+1} = a(-1)^r D_\lambda^{2r+1}\lambda^{2n+1}$,

$$.\supset.\, a(-1)^r[f^{(2r+1)}(x+\lambda)+f^{(2r+1)}(x-\lambda)] = b[f(x+\mu)-f(x-\mu)] ;$$

(51) $a\cos\lambda x = b$ and $\sin\lambda x = 0 \,.\supset.\, af(x+\varphi(\lambda, 1)) = bf(x)$.

It is convenient now to introduce a few definitions. For a particular general choice of the units $e_2, e_3, \cdots$, call $(e_2, e_3, \cdots, e_r)$ the *signature* of

$$f(U_p+e_2u_2+e_3u_3+\cdots+e_ru_r) ,$$

in which U_p involves no symbol $e, e_j, e_k', e_s'', \cdots$ of a unit. Denote by Σ_σ a summation with respect to all possible values of the units occurring in the signature of the function following Σ_σ. If in the signature there are precisely n units, Σ_σ consists of 2^n terms.

With U_r, V_s as before (§2), we have now the important formulas, with $c=1$, or -1,

(52) $$f(x+c\varphi(U_r)) = \Sigma_\sigma f(x+c(u_1+e_2u_2+e_3u_3+\cdots+e_ru_r)) ,$$

(53) $$f(x+c\psi(V_s)) = \Sigma_\sigma e_2e_3\cdots e_s f(x+c(v_1+e_2v_2+\cdots+e_sv_s)) ,$$

(54) $$f(x+c\chi(U_r\,;\,V_s')) = \Sigma_\sigma e_1'e_2'\cdots e_s' f(x+c(u_1+e_2u_2+\cdots+e_ru_r + e_1'v_1'+\cdots+e_s'v_s')) .$$

These are evident by inspection.

The multipliers $e_2\,e_3\cdots e_s$, $e_1'e_2'\cdots e_s'$ on the right of (53), (54) are called the respective *signs* of the particular general terms of the summations, *and also of the respective functions on the left.* In (52) we define the *sign* to be $+1$. The *trace* of each function on the *left* of (52)-(54), *for all values of* x, is that part of the argument of the f on the *right* which contains only $u_j (j=1, \cdots, r)$ and $v_k (k=1, \cdots, s)$, or one of these sets if the other is absent. *In particular, then,* $\varphi_n(U_r)$, $\psi_n(V_s)$, $\chi_n(U_r\,;\,V_s')$ *have the same respective traces and signs as the functions in* (52)-(54). We shall denote the trace of F by $T(F)$, and its sign by $S(F)$. Clearly $S(F_1F_2)=S(F_1)\,S(F_2)$. When several functions $F, F_1, F_2, \cdots$ involving units occur, *the units are treated as umbrae, so that, in particular, units arising in any formula from different functions are to be denoted by different systems of letters, thus* $e_i, e_k', e_l'', \cdots$. The necessity for this convention will be obvious from (55).

Let now each of $\lambda, \mu, \cdots, \nu$ denote a definite one of φ, ψ, χ for any systems $U_a, U_b, \cdots, V_r, V_s, \cdots$ of arguments $u_j, u_k', \cdots, v_l, v_m', \cdots$ $(j, k, \cdots, l, m, \cdots = 1, 2, \cdots)$. Then we have the fundamental theorem

(55) $$f(x+\lambda+\mu+\cdots+\nu) = \Sigma_\sigma S(\lambda\mu\cdots\nu) f(x+T(\lambda)+T(\mu)+\cdots+T(\nu)) .$$

To illustrate the meaning of this for a particular case, let

$$\lambda_n \equiv \varphi_n(U_r) , \quad \mu_n \equiv \psi_n(V_s) , \quad \nu_n \equiv \chi_n(U_p' ; V_t') .$$

Then, remembering the convention about units, we write the traces

$$T(\lambda) \equiv u_1 + e_2 u_2 + \cdots + e_r u_r , \qquad T(\mu) \equiv v_1 + e_2' v_2 + \cdots + e_s' v_s ,$$

$$T(\nu) \equiv u_1' + e_2'' u_2' + \cdots + e_p'' u_p' + e_1''' v_1' + e_2''' v_2' + \cdots + e_t''' v_t''' ,$$

getting for $S(\lambda)$, $S(\mu)$, $S(\nu)$ respectively the values

$$1, \quad e_2' e_3' \cdots e_s' , \quad e_1''' e_2''' \cdots e_t''' .$$

Hence the sign is here $e_2' e_3' \cdots e_s' e_1''' e_2''' \cdots e_t'''$, and the signature is $\Sigma_2^r e_i + \Sigma_2^s e_k' + \Sigma_2^p e_l'' + \Sigma_1^t e_m'''$, so that Σ_σ consists of $2^{r+s+t+p-3}$ terms.

The concept of umbral equality applied to (55) leads to the simplest and most powerful process of the whole algebra. Let ξ, ζ be umbrae for which

$$\xi^n \equiv \xi_n = \zeta_n \equiv \zeta^n \qquad (n = 0 , 1 , 2 , \cdots) .$$

Then we say that *the umbrae* ξ, ζ *are equal*, $\xi = \zeta$, and evidently in any theorem an umbra may be replaced by any other which is equal to it. Hence from (55) we have

$$\tau = \lambda + \mu + \cdots + \nu \tag{56}$$

$$. \supset . f(x+\tau) = \Sigma_\sigma S(\lambda\mu \cdots \nu) f(x + T(\lambda) + T(\mu) + \cdots + T(\nu)) .$$

Although this is obvious it enables us in the applications to dispense with masses of algebraic computation and to reach in one step results for sequences of special functions which hitherto have been attained only by laborious reductions. It is evidently but a special case of the obvious implication

$$\xi = \zeta . \supset . f(x+\xi) = f(x+\zeta) . \tag{57}$$

Let next $f(x)$ be a power series* in x, or a function which can be expanded into such a series. Then, provided the series for $f(x)$, $f(x+\lambda)$, $\cdots$ are convergent for some $|x| > 0$, all the results of this section remain valid for the new interpretation of $f(x)$.

If the series $f(x)$, $f(x+\lambda)$, $\cdots$ are divergent, there still is an extremely useful interpretation of all the formulas of this section. This method of

* When $f(x)$ is a Laurent expansion the algebra is also applicable with a few obvious restatements of theorems to include both positive and negative ranks of coefficients. A detailed example of the algebra in this case is given in the paper, cited in the introduction, on Spitzer's functions.

interpretation was fully discussed in the paper on *Euler algebra* mentioned in the introduction. Briefly, the interpretation merely states the equality of the particular general coefficients in the series obtained by operations in a field from any given set of series, the addition, multiplication, division, subtraction of the field being defined in a specific way, for which see the paper cited. The equality in question can be independently established by means of only the finite processes which we have already discussed. The advantage, however, of using infinite processes is indisputable, and consists in the automatic presentation of valuable transformations of finite formulas which otherwise would probably be overlooked.

Finally it is clear from the way in which they were derived that all results in this section remain true when x, the constants a, b, and all arguments occurring in f are interpreted as umbrae. In such interpretations the usual precautions regarding umbral derivatives must be observed. Thus $D_\lambda \sin \lambda x = \lambda \cos \lambda x$, and the *umbral* multiplication $\lambda \times \cos \lambda x$, must be performed *before* degradation of exponents (passage to ordinaries).

8. The connection of §§ 2-7 with sequences is made as follows: Let $u_n, v_n, \cdots, w_n (n=0, 1, 2, \cdots)$ be any sequences of numbers (in which case n is the only independent variable) or functions (when n is not the only variable), and let each of $\lambda, \mu, \cdots, \nu$ be a definite one of φ, ψ, χ with umbral arguments chosen from among $u, v, \cdots, w$. Then $\lambda_n, \mu_n, \cdots, \nu_n$ $(n=0, 1, 2, \cdots)$ are new sequences of numbers or functions, and the algebra gives the means of rapidly determining their interrelations with the original set of sequences.

The reason for investigating a set of sequences based upon φ, ψ, χ rather than upon some other functions is pragmatic. This is established by actually applying the method to numerous well known functions and to the discovery of others with equally interesting properties. Some of the possibilities were mentioned in the introduction. And although as a matter of logic success in application is the only possible proof of a pragmatic proposition, we can yet see an obvious reason why the algebra does work. Define $\sum (-1)^n u_{2n+1} t^{2n+1}/(2n+1)!$, $\sum (-1)^n u_{2n} t^{2n}/(2n)!$, where t is a parameter, to be the respective generators of the sequences u_{2n+1}, u_{2n} $(n=0, 1, 2, \cdots)$, and for simplicity suppose that for some $|t|>0$ all generators discussed are absolutely convergent. Then clearly we shall not have exhausted the mutual relationships of the sequences $u_n, v_n, \cdots, w_n$ $(n=0, 1, 2, \cdots)$ until we have fully investigated the *field* generated by their generators (all the elements of this field being themselves generators). In this field the functions φ, ψ, χ with umbral arguments $u, v, \cdots, w$, or φ, ψ, χ functions of these, and so on, are precisely those functions which

are generated by multiplication and division from the original generators (the *basis* of the field). As shown in the paper on Euler algebra there is no necessity to consider the convergence of the elements of this field, so that in all cases the derived elements are significant. Or, as pointed out in the introduction, convergence can be ignored otherwise by operating in a certain algebra with an infinite basis.

Occasionally with well known functions $u_n, v_n, \cdots, w_n$ it is more advantageous to discuss not these but $p_n u_n, q_n v_n, \cdots, r_n w_n$, where $p_n, q_n, \cdots, r_n$ are functions of n alone. Accordingly we first work out the theory for the modified functions, translating only the final formulas, if desired, into terms of $u_n, v_n, \cdots, w_n$. This is the case for example with Spitzer's functions, where the appropriate multiplier is $(-1)^n$. Or again $u_n+p_n, v_n+q_n, \cdots$ may be more amenable than $u_n, v_n, \cdots$. This is so for the usual Bernoullian functions, which we slightly modify. Our functions can readily be expressed in terms of others in the literature by means of §14.

PART II. THE FUNCTIONS $\beta, \gamma, \eta, \rho$

9. Only sufficient need be given to illustrate the application of some of the chief processes of Part I to the Bernoullian functions $\beta_n(u)$, $\gamma_n(u)$ and the Eulerian $\eta_n(u)$, $\rho_n(u)$, and to suggest that a systematic treatment of $\beta, \gamma, \eta, \rho$ from this point of view should prove profitable.

The even suffix notation is used for the numbers B, G, E, R of Bernoulli, Genocchi, Euler and Lucas, so that with the exceptions $B_1=-1/2$, $G_1=1$, all the numbers of odd ranks vanish, and the first seven values are

$$\begin{array}{lccccccc} n=0 & 1 & 2 & 3 & 4 & 5 & 6, \\ B_n=1 & -\frac{1}{2} & \frac{1}{6} & 0 & -\frac{1}{30} & 0 & \frac{1}{42}, \\ G_n=0 & 1 & -1 & 0 & 1 & 0 & -3, \\ E_n=1 & 0 & -1 & 0 & 5 & 0 & -61, \\ R_n=\frac{1}{2} & 0 & -\frac{1}{6} & 0 & \frac{7}{30} & 0 & -\frac{31}{42}. \end{array}$$

These will be found useful in checking all formulas given later. The symbolic generators, equivalent to definitions, are

$$(58) \qquad \begin{array}{ll} x\cot x=\cos 2Bx\,, & -x=\sin 2Bx\,, \\ 2x\tan x=\cos 2Gx\,, & 2x=\sin 2Gx\,, \\ \sec x=\cos Ex\,, & 0=\sin Ex\,, \\ x\csc x=2\cos Rx\,, & 0=\sin Rx\,. \end{array}$$

The first column contains the definitions of Lucas, the second is a new and essential detail in the efficient application of the algebra.

In illustration now of the remarks in §8 we may ask what are the relations of the sequences B_n, G_n, E_n, R_n to *any other sequences whatever*,

$u_n, v_n, w_n, \cdots$ $(n=0, 1, 2, \cdots)$? Confining ourselves to the algebra of order 2 we may completely answer the question by combining the symbolic generators soc $u\,x$, soc $v\,x$, soc $w\,x, \cdots$, where "soc" is an abbreviation for "sin or cos," with the definitions (58), and this immediately reduces the investigation to that of functions φ, ψ, χ with umbral arguments chosen from among $2B, 2G, E, R, u, v, w, \cdots$, and to φ, ψ, χ functions of these functions. Thus the solution is contained in the algebra constructed in Part I.

The simplest possibility is that of a single umbra u, and the special case of this in which $u_n=1$ $(n=0, 1, 2, \cdots)$, so that the sequence is $1, 1, 1, \cdots$, gives the theory of the relations between the numbers B, G, E, R. The next simplest specialization for a single umbra u is that in which $u_n \equiv u^n$, u an ordinary, giving the theory of relations between the ordinary Bernoullian and Eulerian functions. Unless otherwise noted we shall take u ordinary.

We can choose as fundamental any functions $\lambda(u, A)$ in which λ is φ or ψ and A is $2B$, $2G$, E or R. Taking $\lambda=\varphi$ we define the four fundamental functions

$$\beta_n(u)=\varphi_n(u, 2B),\ \gamma_n(u)=\varphi_n(u, 2G),\ \eta_n(u)=\rho_n(u, E),\ \rho_n(u)=\varphi_n(u, R),$$

whose symbolic generating identities are therefore (cf. (58))

$$\text{soc}\ \beta(u)x=2x\cot x\ \text{soc}\ ux\equiv 2\cos 2Bx\ \text{soc}\ ux\ , \tag{59}$$

$$\text{soc}\ \gamma(u)x=4x\tan x\ \text{soc}\ ux\equiv 2\cos 2Gx\ \text{soc}\ ux\ , \tag{60}$$

$$\text{soc}\ \eta(u)x=2\sec x\ \text{soc}\ ux\equiv 2\cos Ex\ \text{soc}\ ux\ , \tag{61}$$

$$\text{soc}\ \rho(u)x=x\csc x\ \text{soc}\ ux\equiv 2\cos Rx\ \text{soc}\ ux\ , \tag{62}$$

and whose non-symbolic forms are

$$\beta_n(u)=2\sum_{r=0}2^{2r}\binom{n}{2r}B_{2r}u^{n-2r}\ ,$$

$$\eta_n(u)=2\sum_{r=0}\binom{n}{2r}E_{2r}u^{n-2r}\ ,$$

with $\gamma_n(u)$ written down from $\beta_n(u)$ by replacing B by G, and $\rho_n(u)$ from $\eta_n(u)$ by replacing E by R. We have also from (4), (59)-(62) the equivalent identities

$$\sin\beta(u)x=\sin\chi(2B;\,u)x\ ,\quad \cos\beta(u)x=\cos\varphi(2B,\,u)x\ , \tag{63}$$

$$\sin\gamma(u)x=\sin\chi(2G;\,u)x\ ,\quad \cos\gamma(u)x=\cos\varphi(2G,\,u)x\ , \tag{64}$$

$$\sin \eta(u)x=\sin \chi(E;\, u)x\,, \qquad \cos \eta(u)x=\cos \varphi(E,\, u)x\,, \tag{65}$$

$$\sin \rho(u)x=\sin \chi(R;\, u)x\,, \qquad \cos \rho(u)x=\cos \varphi(R,\, u)x\,. \tag{66}$$

These with (4) give the best set of definitions for the application of Part I to $\beta, \gamma, \eta, \rho$. A complete discussion of these functions presupposes the like for the numbers B, G, E, R, viz., the special case already mentioned. Here it will be sufficient to derive such relations as are needed for purposes of illustration.

We restricted u to be an ordinary. But all that follows holds also when u is an umbra, and similarly for the variables $x, v, w, \cdots$. For definiteness, however, in reading formulas, $x, u, v, w, \cdots$ may be interpreted as their special cases, ordinaries. The more general interpretation, however, is a necessity, as will be seen, in the steps intermediate between an ordinary formula and its transforms.

10. Instead of the functions $\lambda_n(u)$ $(\lambda=\beta, \gamma, \eta, \rho)$ of argument u we can treat in precisely the same way the functions $\lambda_n(g(u))$ having as arguments *any* function $g(u)$ of u, by replacing in (59)-(66) u by $g(u)$. For example, if u is a positive integer, the functions in which $g(u)=2u$ or $2u-1$ are among those most frequently appearing in connection with sums of like powers of the integers $1, 2, \cdots, n$ and similar problems. For the same $g(u)$, u complex, the $\lambda_n(g(u))$ are but slightly different from the Bernoullian and Eulerian functions as most commonly defined. It is convenient in such cases to define *new* Bernoullian and Eulerian functions $\Lambda_n(u)$ *of argument* u; thus

$$\Lambda_n(u)\equiv\lambda_n(g(u)), \quad (\Lambda, \lambda)\equiv(B, \beta), \quad (G, \gamma), \quad (E, \eta), \quad (R, \rho)\,.$$

There need be no confusion between $B_n(u)$ and B_n if the former never is written without the argument u, and so for the rest. The definitions now become

$$\operatorname{soc} B(u)x=\operatorname{soc} \beta(g(u))x=2\cos 2Bx \operatorname{soc} g(u)x\,,$$

and so for the others, all being given by

$$\operatorname{soc} \Lambda(u)x=\operatorname{soc} \lambda(g(u))x\,. \tag{67}$$

We now prove a general theorem regarding transformations of $\Lambda_n(u)$.

Jacobi proved the important result for the common Bernoullian functions, in which $g(u)=2u-1$, that they are invariant to within sign when u is replaced by $1-u$, and Raabe did the like for the common Eulerian functions. Generalizing this to any $g(u)$ we see from (67) that the necessary

and sufficient condition that $\Lambda_n(u)$ $(n=0, 1, 2, \cdots)$ be invariant to within sign when u is replaced by $h(u)$ is that $g(h(u))=\pm g(u)$; in particular

$$g(h(u))=g(u) \,.\supset.\, \Lambda_n(h(u))=\Lambda_n(u) ,$$

$$g(h(u))=-g(u) \,.\supset.\, \Lambda_n(h(u))=(-1)^n\Lambda_n(u) .$$

Thus the automorphic transformations of $\Lambda_n(u)$ $(n=0, 1, 2, \cdots)$ are identical with those of $g(u)$.

As an example, excluding the trivial case $qa-pb=0$, we have

$$\Lambda(u')\equiv\lambda_n\left(\frac{au'+b}{pu'+q}\right)=(-1)^n\lambda_n\left(\frac{au+b}{pu+q}\right)\equiv(-1)^n\Lambda_n(u) ,$$

where

$$u'=-\frac{(qa+pb)u+2qb}{2pau+(qa+pb)} ;$$

and hence for $p=0$, $q=1$, $\lambda_n(au+b)$ is transformed into $(-1)^n\lambda_n(au+b)$ by the substitution $\{u, -u-2b/a\}$, so that in the further special case $(a, b)=(2, -1)$, $\lambda_n(2u-1)$ is transformed into $(-1)^n\lambda_n(2u-1)$ by $\{u, 1-u\}$, giving the theorems of Jacobi and Raabe.

11. From $\beta_n(u)=\varphi_n(u, 2B)$ we find, by (10), the last χ_{2n} in the same section, and (17),

$$\beta_n(u+v)=(u+\beta(v))^n=(v+\beta(u))^n ,$$

and hence, what is otherwise evident from the definition of $\beta_n(u)$, the addition theorem

$$f(x+\beta(u+v))=f(x+u+\beta(v))=f(x+v+\beta(u)) . \tag{68}$$

There are several more interesting types, of which one follows. From (59) on replacing u by $u+v$ we get

$$2x \operatorname{soc}\beta(u+v)x=\tan x \operatorname{soc}(\beta(u)+\beta(v))x ,$$

and hence from (60),

$$8x^2 \operatorname{soc}\beta(u+v)x=\operatorname{soc}\gamma(\beta(u)+\beta(v))x ,$$

so that finally by (47), (49) we have

$$f(x+\gamma(\beta(u)+\beta(v)))=-8f''(x+\beta(u+v)) . \tag{69}$$

In the same way

$$(70)\quad f(x+\beta(\gamma(u)+\gamma(v)))=-8f''(x+\gamma(u+v))\ ,$$

$$(71)\quad f(x+\eta(u)+\eta(v)+1)+f(x+\eta(u)+\eta(v)-1)=f(x+\eta(u+v))\ ,$$

$$(72)\quad f(x+\rho(u)+\rho(v)+1)-f(x+\rho(u)+\rho(v)-1)=2f'(x+\rho(u+v))\ .$$

12. The use of the umbral equalities and substitutions is illustrated by the following example. Suppose we wish to develop a function of $\beta(u)+c\beta(v)$, where $c=1$ or -1, in powers of $u+cv$. We have $\beta(u)=\varphi(u, 2B)$, $\beta(v)=\varphi(v, 2B')$, where the identical umbrae B, B' are distinguished in the notation, as they are to occur in a single formula. Hence

$$f(x+\beta(u)+c\beta(v))=f(x+\varphi(u, 2B)+c\varphi(v, 2B'))\ ,$$

and therefore

$$(73)\qquad f(x+\beta(u)+c\beta(v))=\Sigma_\sigma f(x+2e_1B+2e_2B'+u+cv)\ .$$

To illustrate the same principle we shall obtain this in a more suggestive and essentially simpler manner. From $\beta(u)=\varphi(u, 2B)$ we have by (57), (56)

$$f(x+\beta(u))=f(x+\varphi(u, 2B))=\Sigma_\sigma f(x+u+2e_1B)\ ,$$

valid for x umbral. Take $x=x'+c\beta(v)$, and write $x'+u+2e_1B\equiv x''$. Then

$$f(x'+\beta(u)+c\beta(v))=f(x''+\varphi(v, 2B))=\Sigma_\sigma f(x''+v+2e_2B)\ ;$$

whence, on restoring the value of x'' and replacing x' by x, we have (73).

A relation involving functions f of β, γ, η, ρ is called of *order* r provided r is the greatest number of equal or distinct umbrae chosen from those occurring in the argument of any f. Thus (73) is of order 2. A complete system of relations of order r may be written out similarly to the above. By means of the umbral equalities between the numbers B, G, E, R the relations can be transformed systematically in many ways. For example, from the reciprocal symmetry between β, B and γ, G, evident in the definitions of β, γ, we can write down from (73)

$$(74)\qquad f(x+\beta(u)+c\gamma(v))=\Sigma_\sigma f(x+2e_1B+2e_2G+u+cv)\ .$$

Now, if we had reproduced here the preliminary algebra of B, G, E, R necessary for a complete discussion of β, γ, η, ρ, we should have seen immediately that the right of (74) is at once reducible, on account of the

presence of B, G together in a linear combination. As it is, the first pair in (58) serve the same purpose, giving by multiplication

$$\cos\varphi(2B, 2G)x = 4x^2 = \cos\psi(2B, 2G) ,$$

and hence from (49) with $\lambda=0$, $r=1$,

$$-8f''(x)=f(x+\varphi(2B, 2G))+f(x-\varphi(2B, 2G)) ,$$
$$-8f''(x)=f(x+\psi(2B, 2G))+f(x-\psi(2B, 2G)) .$$

From these by subtraction we get

$$f(x+2B-2G)+f(x-2B+2G)=0 ,$$

and hence, combining this with the result of addition,

$$-8f''(x)=\Sigma_e f(x+2e_1B+2e_2G) . \tag{75}$$

Apply this to the right of (74). Then

$$f(x+\beta(u)+c\gamma(v))=-8f''(x+u+cv) . \tag{76}$$

In a complete exposition the derivation of (76) would be still shorter, for we should then have available all theorems such as (75) on the numbers B, G, E, R. By repetition of (76) we find a curious result. Replace u by $\beta(u)$, v by $\gamma(v)$, repeat the process $r-1$ times, using (76) each time to reduce the new right hand member to a form free from β, γ. Write $\beta_n(\beta(\cdots\beta(u)\cdots))$ (r symbols β) $\equiv \beta_n^{[r]}(u)$, and similarly for γ. Then

$$f(x+\beta^{[r]}(u)+c\gamma^{[r]}(v))=(-8)^r f^{[2r]}(x+u+cv) . \tag{77}$$

In the same way, or as noted in a moment, we see the following interesting reciprocities between pairs of β, γ, η, ρ ($c=\pm1$):

$$f(x+\beta(\gamma(u)))=f(x+\gamma(\beta(u)))=-8f''(x+u) , \tag{78}$$

$$f(x+\beta(\eta(u)))=f(x+\eta(\beta(u)))=4f(x+\rho(u)) , \tag{79}$$

$$f(x+c\beta(\rho(u))+1)-f(x+c\beta(\rho(u))-1)=2f'(x+c\beta(u)) , \tag{80}$$

and the like with β, ρ interchanged only on the left;

$$f(x+\gamma(\eta(u))+1)+f(x+\gamma(\eta(u))-1)=4f(x+\nu(u)) , \tag{81}$$

and the like with γ, η interchanged only on the left;

$$f(x+\gamma(\rho(u)))=f(x+\rho(\gamma(u)))=f(x+\eta(u)) , \tag{82}$$

$$f(x+\tfrac{1}{2}\eta(\rho(2u)))=f(x+\tfrac{1}{2}\rho(\eta(2u)))=f(x+\rho(u)) . \tag{83}$$

The alternative derivation is from (59)-(62). Thus, for example, (59), (60) give at once

$$\operatorname{soc} \beta(\eta(u))x = \operatorname{soc} \eta(\beta(u))x = 4 \operatorname{soc} \rho(u)x ,$$

and therefore (79).

Many of the theorems of the types obtainable by the methods of this section immediately suggest investigations worth detailed development. For instance the presence of the identical umbrae B, B' in (73) directs us to introduce into the theory of the Bernoullian and allied *functions* the Bernoullian and allied *numbers of higher order* which were first defined by Lucas. Thus the right of (73) need involve only the single umbra $B^{(2)}$ of the Bernoullian numbers of the second order. This in turn suggests that we generalize the whole theory of the Bernoullian and Eulerian functions by introducing such functions of order $r \geqq 1$, and so obtain the theory of the ordinary Bernoullian and Eulerian functions by taking $r = 1$ in the results. The general theory is as readily developed by means of the algebra in Part I as is its special case $r = 1$.

13. We take space merely to indicate three of the far reaching applications of relations such as those in the preceding section. First, an indefinite number of curtate relations between functions β, γ, η, ρ are obtainable by taking for $f(x)$ the product of any number of polynomials in x of equal or unequal degrees.

Second, the relations between functions β, γ, η, ρ whose ranks are the successive numbers of any arithmetic progression with first term and common difference positive integers are found uniformly as follows. Simson's well known method for summing the series obtained by selecting from any given series every mth term leads at once, through the functions of Olivier and a corresponding generalization of the symmetric functions of Kronecker, to the algebra of order m (cf. §1), when for the given series we take the expansion of an exponential, circular or hyperbolic function. Multiplying the nth term of the new series by u_n or v_n, or $w_n, \cdots$, we obtain precisely as in Part I a set of umbral generators from which we construct as there the algebra of these generalized umbral series and cosines. Clearly if $u_n = v_n = w_n \cdots = 1$ $(n = 0, 1, 2, \cdots)$, the result of multiplying any series by a so specialized umbral sine or cosine of the new type selects from the series all terms whose ranks are in arithmetic progression with common difference m. For other sequences $u_n, v_n, w_n, \cdots$, the selected series are a generalization of these, and lead to an extension in a wholly new direction of the Bernoullian and Eulerian functions. The entire theory of the relations between the coefficients of the selected series is isomorphic to that

of the indicated generalization of Olivier's functions, and is obtained with great ease by means of the algebra of order m. The case $m=2$ developed in Part I is a complete guide to $m>2$.

Third, the application of all that precedes to the special case in which each of the variables $u_i, v_k, \cdots$ is an integer is of importance in the theory of numbers, as it leads to many congruences for the numbers B, G, E, R.

For rational arguments the functions $\beta, \gamma, \eta, \rho$ yield the theory of sums of like powers of integers in arithmetic progression, or such sums with periodic successions of $\pm$ signs. Thus it is easily seen that if u, a, b are positive integers, and

$$b\sigma_n(u)\equiv 2\sum_{j=0}^{u}(a+bj)^n, \quad bv\equiv 2a+(2u+1)b, \quad bw\equiv 2a-b\ ,$$

then (from the expansion of $\sum_0^u 2xp(a+bj)x$),

$$f(x+\rho(v))-f(x+\rho(w))=2f'(x+\sigma(u))\ .$$

14. For comparison with other Bernoullian functions in the literature we add the exponential definitions of the generators. These are found from (59)-(62) or from (63)-(66) by writing down the values of

$$\cos\{\lambda(u)x/i\}+i\sin\{\lambda(u)x/i\}\ \ (\lambda=\beta,\ \gamma,\ \eta,\ \rho)\ :$$

$$e^{\beta(u)x}=2xe^{ux}\left(\frac{e^{2x}+1}{e^{2x}-1}\right), \qquad e^{\gamma(u)x}=-4xe^{ux}\left(\frac{e^{2x}-1}{e^{2x}+1}\right),$$

$$e^{\eta(u)x}=4\ e^{ux}\left(\frac{e^{x}}{e^{2x}+1}\right), \qquad e^{\rho(u)x}=\ 2xe^{ux}\left(\frac{e^{x}}{e^{2x}-1}\right)\ .$$

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BUNDLES AND PENCILS OF NETS ON A SURFACE*

BY

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I. INTRODUCTION

A *net* of curves on a surface consists of two one-parameter families of curves such that through each point of the portion of the surface under consideration there passes just one curve of each family, the two tangents at each point being distinct. It is a characteristic property of *conjugate nets* that at every surface point the two tangents of every conjugate net separate the tangents of the asymptotic net harmonically. In other words, at every point of a surface the cross ratio of the two asymptotic tangents and any two conjugate tangents is -1. Recent developments in projective differential geometry seem to indicate that these ideas should be generalized.

In the first place, the number -1 may be replaced by any number r, and we may then consider the class of all nets on a surface, every one of which has the property that at every surface point its tangents form with the asymptotic tangents the constant cross ratio r. In the second place, the asymptotic net may be replaced by an arbitrary fundamental net. We are thus led to introduce the following definition. *The class of all nets on a surface, every one of which has the property that at every surface point its two tangents form with the two tangents of a fundamental net the same cross ratio, will be called a bundle of nets.* A bundle of nets on a surface is therefore determined by a fundamental net and a number r. For example, all conjugate nets on a surface constitute a bundle, for which the fundamental net is the asymptotic net, and for which $r=-1$.

The one-parameter family of conjugate nets on a surface, each of which has the property that, at every surface point, its tangents form with the tangents of a fundamental conjugate net a constant cross ratio, has been called† by Wilczynski a *pencil of conjugate nets*. This definition can be extended from the bundle of conjugate nets to any bundle of nets. *A pencil of nets in a bundle is a one-parameter family of nets in the bundle, each of which has the property that, at every surface point, its two tangents form with the two tangents of a fundamental net of the bundle a constant cross ratio.* This cross ratio is different for different nets of the pencil, but for a given

* Presented to the Society, December 26, 1924; received by the editors in June, 1925.

† Wilczynski, American Journal of Mathematics, vol. 42 (1920), p. 216.

net is the same at every point of the surface. A pencil of nets in a bundle is determined by a fundamental net of the bundle.

II. Bundles of nets

In order to formulate analytically the definitions which we have introduced, let us suppose that the four homogeneous coördinates $y^{(1)}, \cdots, y^{(4)}$ of an arbitrary point on a surface S are given as analytic functions of two independent variables u, v. Then a net of curves on S may be represented by a differential equation of the form

$$A\,du^2 + 2B\,du\,dv + C\,dv^2 = 0, \tag{1}$$

wherein A, B, C are functions of u, v such that $AC - B^2 \neq 0$. A fundamental net N_1 being given by equation (1), let us consider a second net N_2 defined by the equation

$$L\,du^2 + 2M\,du\,dv + N\,dv^2 = 0, \quad LN - M^2 \neq 0. \tag{2}$$

We wish to impose on the net N_2 the condition that at every surface point its two tangents shall form with the two tangents of the net N_1 a constant cross ratio. So we write down the condition that the two roots of equation (2), regarded as a quadratic in the ratio $dv : du$, may form with the two roots of equation (1) a cross ratio r which is independent of u, v. This condition reduces to

$$(1-r)^2\,(AN - 2BM + CL)^2 = 4(1+r)^2\,(AC - B^2)\,(LN - M^2). \tag{3}$$

The class of all nets represented on a surface S by an equation of the form (2), *whose coefficients satisfy equation* (3) *with given functions A, B, C and a fixed value of r, is a bundle of nets.* A bundle of nets on a surface depends therefore on one arbitrary function, and is determined by a net N_1 and a number r.

The symmetry of equation (3) shows that if a net N_2 belongs to the bundle determined by a net N_1 and a constant cross ratio r, then N_1 belongs to the bundle determined by N_2 and the same constant r. Interchanging the two component families of N_1, or of N_2, changes r to $1/r$, but leaves equation (3) invariant. If $r = -1$, this equation simplifies into the well known condition that at every surface point the two tangents of N_2 separate the two tangents of N_1 harmonically: namely, the vanishing of the harmonic invariant of the binary quadratic differential forms in equations (1) and (2).

If N_1 is taken for the parametric net on S, then $A = C = 0$, $B \neq 0$, and equation (3) takes the form

$$4\,rM^2 = (1+r)^2 LN. \tag{4}$$

We wish to find the differential equation of the most general net N_λ that belongs to the bundle B_r determined by the parametric net and a constant cross ratio r. To this end let us consider an arbitrary one-parameter family F_λ of curves on S, distinct from the parametric curves. The equation of such a family F_λ may be written in the form

$$dv-\lambda\, du=0, \tag{5}$$

wherein λ is an arbitrary function of u, v. And let us seek to determine the most general family of curves, represented by the equation

$$dv-\mu\, du=0,$$

which forms with F_λ a net of the required kind. Multiplying together the left members of the last two equations and comparing the result with equation (2), we find

$$L=\lambda\mu, \quad 2M=-\lambda-\mu, \quad N=1.$$

Substituting these functions into equation (4), and solving for μ, we find that we may take $\mu=r\lambda$. *Therefore, if the family F_λ, which is represented by equation* (5), *and the family $F_{r\lambda}$, which is represented by the equation*

$$dv-r\lambda\, du=0, \tag{6}$$

are taken together to form the net N_λ, whose differential equation is

$$(dv-\lambda\, du)\,(dv-r\,\lambda\, du)=0, \tag{7}$$

then N_λ is the most general net that belongs to the bundle B_r determined by the parametric net on any surface S and a constant cross ratio r.

We might equally well have taken $\mu=\lambda/r$, but this choice is equivalent to interchanging the first and second component families of the net N_λ.

III. A PENCIL OF NETS IN A BUNDLE

We propose now to obtain the differential equation of the most general net of the pencil determined by the net N_λ in the bundle B_r, and then to find the differential equation of all the curves of this pencil.

The equation of an arbitrary net of the bundle B_r may be written in the form

$$(dv-\mu\, du)\,(dv-r\,\mu\, du)=0. \tag{8}$$

We wish to determine the function μ so that, at every surface point, the two tangents of this net may form with the two tangents of the net N_λ a constant cross ratio c. Multiplying out the left members of each of equations (7) and (8), and substituting the resulting coefficients into equation (3),

we find that μ must be proportional to λ, so that we have $\mu=\lambda h$, where h is a constant depending on c for its value. *Therefore the most general net $N_{\lambda\lambda}$ of the pencil determined by the net N_λ in the bundle B_r,—which is itself determined by the parametric net on any surface S and a constant cross ratio r,—is represented by the equation*

$$(dv-\lambda\, h\, du)\,(dv-r\lambda\, h\, du)=0, \tag{9}$$

wherein h is an arbitrary constant. If any other net of this pencil than the net N_λ is selected as the fundamental net to determine a pencil of nets in the bundle B_r, the same pencil is obtained as before. *Therefore a pencil of nets in a bundle is independent of the one of its nets used to determine it.*

The totality of curves contained in all the nets of a pencil in a bundle is a two-parameter family of curves. Such a family is represented in curvilinear coördinates by a differential equation of the second order. If we set either factor of equation (9) equal to zero, solve the resulting equation for h, and differentiate totally with respect to u, we obtain *the differential equation of all the curves contained in the pencil of nets determined by the net N_λ in the bundle B_r, namely*

$$\frac{d^2v}{du^2}-\frac{\lambda_u}{\lambda}\frac{dv}{du}-\frac{\lambda_v}{\lambda}\left(\frac{dv}{du}\right)^2=0. \tag{10}$$

If the parametric net is asymptotic, this equation is identical with the equation which we have found* for the curves of a pencil of conjugate nets.

It will be remarked that, although equation (10) appeared as the differential equation of the curves contained in a pencil of nets in the bundle determined by the parametric net and a constant cross ratio r, still the equation is independent of r. It follows that *the curves of a pencil of nets in a bundle are independent of the cross ratio used in defining the bundle.* The reason why equation (10) is identical in form with the equation of the curves in a pencil of *conjugate* nets thus becomes evident.

The curves $u=$const. are not included among the integral curves of equation (10). If we wish to include them, we may write the equation in a more general and more symmetrical form by supposing that u and v are functions of a parameter t along an integral curve. Equation (10) then becomes

$$u'v''-u''v'-\frac{\lambda_u}{\lambda}u'^2v'-\frac{\lambda_v}{\lambda}u'v'^2=0, \tag{10'}$$

wherein accents denote differentiation with respect to t.

* Lane, *A general theory of conjugate nets*, these Transactions, vol. 23 (1922), p. 289.

IV. Systems G

It is convenient to introduce here the notion of a *system G* of curves on a surface, by means of the following definition. *A system G of curves on a surface S consists of three distinct one-parameter families of curves together with the curves of all the one-parameter families each of which has the property that, at every surface point, its tangent and the three tangents belonging to the three fundamental families form a constant cross ratio.* If two of the fundamental families of a system G constitute the parametric net on S, and if the third family is F_λ, represented by equation (5), then the family $F_{r\lambda}$, represented by equation (6), is a family of curves belonging to the system G. Therefore all the curves of the system G determined by the parametric net and F_λ are integral curves of equation (10′). Conversely, every equation of the form (10′) can be integrated once and its integral curves can thus be shown to constitute a system G.

If we select any three distinct one-parameter families of curves contained in a system G to determine another system G, we find the same system G again. *Therefore a system G is independent of the three of its one-parameter families used to determine it.*

The curves of a system G constitute a special family of hypergeodesics.* In fact, a family of hypergeodesics can be shown to be a system G if, and only if, the curves of the family can be assembled into a single infinity of one-parameter families of curves such that, when any four of the families are selected, the cross ratio, at every surface point, of the four tangents belonging to these four families is constant.

The single infinity of one-parameter families of curves contained in a system G may be paired so as to form a one-parameter family of nets in various ways. We have seen the families of integral curves equation (10) paired, for example, as in equation (9) to form the nets belonging to a pencil of nets in the bundle B_r. And in particular, if the parametric net is asymptotic and if $r=-1$, we can pair the families of integral curves of (10) to form the conjugate nets of the pencil which is determined by the conjugate net

$$dv^2-\lambda^2du^2=0. \tag{11}$$

Any one of these nets can have its equation written in the form

$$dv^2-\lambda^2h^2du^2=0. \tag{12}$$

* Wilczynski, *Some generalizations of geodesics*, these Transactions, vol. 23 (1922), p. 226.

we find that μ must be proportional to λ, so that we have $\mu=\lambda h$, where h is a constant depending on c for its value. *Therefore the most general net $N_{\lambda\lambda}$ of the pencil determined by the net N_λ in the bundle B_r,—which is itself determined by the parametric net on any surface S and a constant cross ratio r,—is represented by the equation*

$$(dv-\lambda\, h\, du)\,(dv-r\lambda\, h\, du)=0, \tag{9}$$

wherein h is an arbitrary constant. If any other net of this pencil than the net N_λ is selected as the fundamental net to determine a pencil of nets in the bundle B_r, the same pencil is obtained as before. *Therefore a pencil of nets in a bundle is independent of the one of its nets used to determine it.*

The totality of curves contained in all the nets of a pencil in a bundle is a two-parameter family of curves. Such a family is represented in curvilinear coördinates by a differential equation of the second order. If we set either factor of equation (9) equal to zero, solve the resulting equation for h, and differentiate totally with respect to u, we obtain *the differential equation of all the curves contained in the pencil of nets determined by the net N_λ in the bundle B_r, namely*

$$\frac{d^2v}{du^2}-\frac{\lambda_u}{\lambda}\frac{dv}{du}-\frac{\lambda_v}{\lambda}\left(\frac{dv}{du}\right)^2=0. \tag{10}$$

If the parametric net is asymptotic, this equation is identical with the equation which we have found* for the curves of a pencil of conjugate nets.

It will be remarked that, although equation (10) appeared as the differential equation of the curves contained in a pencil of nets in the bundle determined by the parametric net and a constant cross ratio r, still the equation is independent of r. It follows that *the curves of a pencil of nets in a bundle are independent of the cross ratio used in defining the bundle.* The reason why equation (10) is identical in form with the equation of the curves in a pencil of *conjugate* nets thus becomes evident.

The curves $u=$const. are not included among the integral curves of equation (10). If we wish to include them, we may write the equation in a more general and more symmetrical form by supposing that u and v are functions of a parameter t along an integral curve. Equation (10) then becomes

$$u'v''-u''v'-\frac{\lambda_u}{\lambda}u'^2v'-\frac{\lambda_v}{\lambda}u'v'^2=0, \tag{10'}$$

wherein accents denote differentiation with respect to t.

* Lane, *A general theory of conjugate nets*, these Transactions, vol. 23 (1922), p. 289.

IV. Systems G

It is convenient to introduce here the notion of a *system* G of curves on a surface, by means of the following definition. *A system G of curves on a surface S consists of three distinct one-parameter families of curves together with the curves of all the one-parameter families each of which has the property that, at every surface point, its tangent and the three tangents belonging to the three fundamental families form a constant cross ratio.* If two of the fundamental families of a system G constitute the parametric net on S, and if the third family is F_λ, represented by equation (5), then the family $F_{r\lambda}$, represented by equation (6), is a family of curves belonging to the system G. Therefore all the curves of the system G determined by the parametric net and F_λ are integral curves of equation (10′). Conversely, every equation of the form (10′) can be integrated once and its integral curves can thus be shown to constitute a system G.

If we select any three distinct one-parameter families of curves contained in a system G to determine another system G, we find the same system G again. *Therefore a system G is independent of the three of its one-parameter families used to determine it.*

The curves of a system G constitute a special family of hypergeodesics.* In fact, a family of hypergeodesics can be shown to be a system G if, and only if, the curves of the family can be assembled into a single infinity of one-parameter families of curves such that, when any four of the families are selected, the cross ratio, at every surface point, of the four tangents belonging to these four families is constant.

The single infinity of one-parameter families of curves contained in a system G may be paired so as to form a one-parameter family of nets in various ways. We have seen the families of integral curves equation (10) paired, for example, as in equation (9) to form the nets belonging to a pencil of nets in the bundle B_r. And in particular, if the parametric net is asymptotic and if $r=-1$, we can pair the families of integral curves of (10) to form the conjugate nets of the pencil which is determined by the conjugate net

$$dv^2-\lambda^2du^2=0. \tag{11}$$

Any one of these nets can have its equation written in the form

$$dv^2-\lambda^2h^2du^2=0. \tag{12}$$

* Wilczynski, *Some generalizations of geodesics*, these Transactions, vol. 23 (1922), p. 226.

V. A GENERALIZATION OF CONJUGATE NETS

The bundle of nets determined on a surface by the asymptotic net and an arbitrary constant cross ratio r is a generalization of the bundle of conjugate nets, reducing to the bundle of conjugate nets if $r=-1$. We are now in position to examine the theory of conjugate nets critically with a view to determining what features of this theory are due to the *constancy* of the cross ratio of conjugate and asymptotic tangents, and what features depend also on the *size* of this constant. We shall confine ourselves for the most part, however, to an examination of the *ray-point cubic* and *ray conic*, and shall find that the ray-point cubic depends only on the constancy of the cross ratio, whereas the ray conic depends on the size of this constant as well.

In order to define the ray-point cubic, let us consider a surface S_y on which the asymptotic net is parametric. Such a surface is an integrating surface of a system of differential equations which can be reduced* to the form

$$\begin{aligned} y_{uu}+2by_v+fy&=0,\\ y_{vv}+2a'y_u+gy&=0. \end{aligned} \tag{13}$$

The tangents of the family F_λ represented on S_y by the equation

$$dv-\lambda\, du=0 \tag{5 bis}$$

form a congruence. The two focal surfaces of this congruence are S_y and another surface which we may denote by S_σ. The tangent at a point P_y on S_y of the curve of F_λ through P_y touches S_σ at a point σ_λ which is called the *ray point* of P_y with respect to F_λ. The coördinates of the point σ_λ are given† by the formula

$$\sigma_\lambda=\rho+\lambda\sigma+\frac{1}{\lambda}(a'\lambda^3+b)y, \tag{14}$$

wherein we have placed

$$\rho=y_u+\frac{1}{2}\,\frac{\lambda_u}{\lambda}y,\quad \sigma=y_v-\frac{1}{2}\,\frac{\lambda_v}{\lambda}y. \tag{15}$$

The coördinates of the ray point $\sigma_{r\lambda}$ of P_y with respect to the family $F_{r\lambda}$ represented by equation (6) are obtained by replacing λ by $r\lambda$ in formula (14), with the result

$$\sigma_{r\lambda}=\rho+r\lambda\sigma-\frac{1}{r\lambda}(a'r^3\lambda^3+b)y. \tag{16}$$

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* Wilczynski, First Memoir, these Transactions, vol. 8 (1907), p. 233.

† Green, *Memoir on the general theory of surfaces*, these Transactions, vol. 20 (1919), p. 131.

If we replace λ by λh in formulas (14) and (16), we obtain the ray points $\sigma_{\lambda\lambda}$ and $\sigma_{r\lambda\lambda}$ of the point P_y with respect to the first and second component families of the net $N_{\lambda\lambda}$ represented by equation (9). If we use the points y, ρ, σ in the tangent plane of S_y at P_y as the vertices of a triangle of reference for a local coördinate system, with suitably chosen unit point, the local coördinates of the ray point $\sigma_{\lambda\lambda}$ are given by the following equations:

$$x_1 = -(a'\lambda^3 h^3 + b), \quad x_2 = \lambda h, \quad x_3 = \lambda^2 h^2. \tag{17}$$

Finally, if we eliminate h from these equations and make the result homogeneous in the usual way, we obtain *the equation of the locus of the ray point $\sigma_{\lambda\lambda}$ as the net $N_{\lambda\lambda}$ varies over all the nets of the pencil determined by the net N_λ in the bundle B_r,—which is itself determined by the asymptotic net and a constant cross ratio r,—namely*

$$a'x_3^3 + bx_2^3 + x_1x_2x_3 = 0. \tag{18}$$

The locus of the ray point $\sigma_{r\lambda\lambda}$ is the same curve. This is the ray-point cubic which we set out to define.

It will be observed that the ray-point cubic (18) *is independent of r*, and is therefore the same as the ray-point cubic of the pencil of conjugate nets determined by the conjugate net (11), which we have studied* elsewhere. This cubic is really determined by the surface S_y and the family F_λ. If, however, any other family of the system G determined by the asymptotic net and F_λ is used to determine a ray-point cubic, the same curve is obtained as before. *Therefore there is in the tangent plane of S_y at P_y just one ray-point cubic associated with the system G determined by the asymptotic net and any family F_λ. The ray point of P_y with respect to each family of the system G lies on this curve.* This conclusion can also be reached by use of some of Wilczynski's results concerning hypergeodesics.

We shall next consider the ray conic. The ray of a point P_y with respect to a net is the line joining the ray points of P_y with respect to the two component families of the net. The ray of P_y with respect to the conjugate net (12) has coördinates given by the equations

$$u_1 = \lambda^2 h^2, \quad u_2 = a'\lambda^4 h^4, \quad u_3 = b. \tag{19}$$

Eliminating h, we obtain *the equation in line coördinates of the envelope of the rays of P_y with respect to all the nets of the pencil of conjugate nets determined by the net* (11), *namely*

$$a'bu_1^2 = u_2u_3. \tag{20}$$

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* Lane, *A general theory of conjugate nets*, these Transactions, vol. 23 (1922), p. 290.

This is the ray conic* of the pencil of conjugate nets. The ray (19) touches the ray conic (20) in the point Q whose coördinates are given by the equations

$$x_1 = -2a'b\lambda^2h^2, \quad x_2 = b, \quad x_3 = a'\lambda^4h^4, \tag{21}$$

so that the equation of the ray conic in point coördinates is

$$4a'bx_2x_3 = x_1^2. \tag{22}$$

Observations similar to those made in connection with the ray-point cubic show that *there is in the tangent plane of S_y at P_y just one ray conic associated with the system G determined by the asymptotic net and any family F_λ. The ray of P_y with respect to the conjugate net determined by each family of the system G is tangent to this ray conic.*

In order to study the pencil of conics touching the asymptotic tangents at the points where the ray conic touches these lines, let us consider again the net $N_{\lambda h}$ represented by equation (9). The first family of this net determines the conjugate net (12), and the ray of P_y with respect to this net touches the ray conic (22) at the point Q, whose coördinates are given by equation (21). The second family of the net $N_{\lambda h}$ determines also a conjugate net, whose equation is

$$dv^2 - r^2\lambda^2h^2du^2 = 0. \tag{23}$$

The ray of P_y with respect to this net touches the ray conic (22) at the point Q_r whose coördinates are given by the equations

$$x_1 = -2a'br^2\lambda^2h^2, \quad x_2 = b, \quad x_3 = a'r^4\lambda^4h^4. \tag{24}$$

The line l_r joining the contact points Q and Q_r has coördinates which are given by the equations

$$u_1 = (1+r^2)\lambda^2h^2, \quad u_2 = 2a'r^2\lambda^4h^4, \quad u_3 = 2b. \tag{25}$$

If we eliminate h from these equations and make the result homogeneous, we obtain the equation of the envelope of the line l_r as the net $N_{\lambda h}$ varies over all the nets of the pencil determined by the net N_λ in the bundle B_r, namely

$$4a'br^2u_1^2 = (1+r^2)^2u_2u_3. \tag{26}$$

The line l_r touches this conic at the point whose coördinates are given by the equations

$$x_1 = -4a'br^2\lambda^2h^2, \quad x_2 = b(1+r^2), \quad x_3 = a'r^2(1+r^2)\lambda^4h^4, \tag{27}$$

* Lane, loc. cit., p. 293.

so that the equation of the same conic in point coördinates is

$$16a'br^2x_2x_3=(1+r^2)^2x_1^2. \tag{28}$$

If $r=-1$, so that the bundle B_r is the bundle of conjugate nets, this conic reduces to the ray conic (22). If r is regarded as a parameter, all the conics represented by equation (28) belong to a pencil of conics, every one of which touches each asymptotic tangent at the point where the flex ray* of the ray-point cubic (18) intersects the tangent.

The polar line of the point (27) with respect to the ray conic has coördinates given by the equations

$$u_1=2r^2\lambda^2h^2, \quad u_2=a'r^2(1+r^2)\lambda^4h^4, \quad u_3=b(1+r^2), \tag{29}$$

and the coördinates of the intersection points of this line with the ray conic are found to be given by the following formulas:

$$x_1=-2a'br(1+r^2)\left[2r\pm i(1-r^2)\right]\lambda^2h^2,$$

$$x_2=b\left[-1+6r^2-r^4\pm 4ri(1-r^2)\right],$$

$$x_3=a'r^2(1+r^2)^2\lambda^4h^4,$$

wherein $i=\sqrt{-1}$. Therefore, if we suppose for a moment that all the numbers used hitherto have been real, these intersections are imaginary, so that the point (27) is *inside* the ray conic, and *all the conics represented by equation* (28) *for different values of* r *are inside the ray conic.* Moreover, it can be shown under this assumption that *the conics* (28) *intersect the ray-point cubic in imaginary points only.*

As the net $N_{\lambda h}$ varies over all the nets of the pencil determined by the net N_λ in the bundle B_r, the line (29) envelopes a conic whose equation in line coördinates is

$$a'b(1+r^2)^2u_1^2=4r^2u_2u_3. \tag{30}$$

The point of contact of the line (29) with this conic has coördinates given by the equations

$$x_1=-a'b(1+r^2)\lambda^2h^2, \quad x_2=b, \quad x_3=a'r^2\lambda^4h^4, \tag{31}$$

so that the equation of the same conic in point coördinates is

$$r^2x_1^2=a'b(1+r^2)^2x_2x_3\,. \tag{32}$$

* Lane, loc. cit., p. 290.

Of course, the point (31) is the intersection of the line (19), which is tangent to the ray conic at Q, and the line

$$u_1=r^2\lambda^2h^2, \quad u_2=a'r^4\lambda^4h^4, \quad u_3=b, \tag{33}$$

which is tangent to the ray conic at Q_r. These two lines are the rays of P_y with respect to the two conjugate nets determined by the first and second component families of the net $N_{\lambda h}$. If again we suppose for a moment that all numbers used are real unless otherwise designated, the point (31) is *outside* the ray conic and *all the conics represented by equation* (32) *for different values of* r *are outside the ray conic, but are in the same pencil as the conics* (28). *Moreover, each conic* (32) *can be shown to intersect the ray-point cubic in two real and four imaginary points.*

We shall now show how the ray conic may be generalized in still another way. The ray of P_y with respect to the net $N_{\lambda h}$ joins the ray points $\sigma_{\lambda h}$ and $\sigma_{r\lambda h}$ of P_y with respect to the two component families of the net. The line coördinates of this ray are found, by means of equations (17), to be given by the equations

$$\begin{aligned} u_1&=r\lambda^2h^2, \\ u_2&=\lambda h[-a'r^2\lambda^3h^3+b(1+r)], \\ u_3&=a'r(1+r)\lambda^3h^3-b. \end{aligned} \tag{34}$$

If we eliminate h from these equations and make the result homogeneous, we obtain *the equation, in line coördinates, of the envelope of the ray of* P_y *with respect to an arbitrary net* $N_{\lambda h}$ *of the pencil of nets determined by the net* N_λ *in the bundle* B_r, *as* $N_{\lambda h}$ *varies over all the nets of this pencil, namely,*

$$\begin{aligned} &r^3u_2^2u_3^2+r^2(1+r)^2u_1(a'u_2^3+bu_3^3) \\ &\quad+a'br(1+r+r^2)\,(1+4r+r^2)u_1^2u_2u_3-a'^2b^2(1+r+r^2)^3u_1^4=0. \end{aligned} \tag{35}$$

This is a curve of class four and order six. It has a double point at P_y, *the double point tangents being the asymptotic tangents.*

If r is regarded as a parameter, equation (35) represents a one-parameter family of curves. We shall characterize two particular curves of this family. First, if $1+r=0$, the curve (35) reduces to the ray conic (20) counted twice. Second, if $1+r+r^2=0$, the equation (35) reduces to

$$u_2^2u_3^2+u_1(a'u_2^3+bu_3^3)=0. \tag{36}$$

This curve is the only curve of the family which is tangent to the flex ray of the ray-point cubic (18), ***and it has the flex ray for triple tangent, the contact***

points being the points where the Segre tangents meet the flex ray, namely, the points represented by the equation

$$a'u_2^3+bu_3^3=0.$$

The curve has no inflexions, four double points, and six cusps.

We shall find it profitable to obtain the parametric equations of the envelope (35) in point coördinates. Any point η on the ray of P_y with respect to the net $N_{\lambda\Lambda}$ is given by

$$\eta=\alpha\sigma_{\lambda\Lambda}+\beta\sigma_{r\lambda\Lambda}=(\alpha+\beta)\rho+(\alpha+\beta r)\lambda h\sigma - \frac{1}{r\lambda h}[\alpha r(a'\lambda^3h^3+b)+\beta(a'r^3\lambda^3h^3+b)]y, \tag{37}$$

where α, β are arbitrary functions of u, v. As $N_{\lambda\Lambda}$ varies over all the nets of the pencil, the point η describes a curve. The point $d\eta/dh$ is on the tangent of this curve, and the point η will be the point of contact of the ray with its envelope if, and only if, $d\eta/dh$ is a linear combination of $\sigma_{\lambda\Lambda}$ and $\sigma_{r\lambda\Lambda}$. Imposing this condition on η, we find

$$\alpha : \beta=(a'r^2\lambda^3h^3+b) : (a'r\lambda^3h^3+b).$$

Using this ratio in equations (37), we find that the point of contact of the ray with its envelope has local coördinates given by

$$\begin{aligned} x_1&=-b^2(1+r)-2a'br(1+r^2)\lambda^3h^3-a'^2r^3(1+r)\lambda^6h^6,\\ x_2&=r\lambda h[2b+a'r(1+r)\lambda^3h^3],\\ x_3&=r\lambda^2h^2[b(1+r)+2a'r^2\lambda^3h^3]. \end{aligned} \tag{38}$$

These are the parametric equations of the envelope in terms of the parameter h.

VI. The Segre-Darboux system G

The Segre curves, whose differential equation is

$$a'dv^3-b\,du^3=0,$$

and the Darboux curves, whose equation is

$$a'dv^3+b\,du^3=0,$$

play a significant rôle in the projective differential geometry of a curved surface. The six one-parameter families of Segre and Darboux curves belong to a system G, which also contains the asymptotic net, and which we shall call the Segre-Darboux system G.

We shall now establish two theorems concerning the Segre-Darboux system G. The ray of P_y with respect to the net N_λ, represented by equa-

tion (7), joins the ray point σ_λ, given by (14), and the ray point $\sigma_{r\lambda}$, given by (16). The point of contact η of this ray with its envelope (35) is given by

$$\eta=(a'r^2\lambda^3+b)\sigma_\lambda+(a'r\lambda^3+b)\sigma_{r\lambda}.$$

If the family F_λ is any one of the three families of the Segre-Darboux system G for which the differential equation is

$$a'r^2dv^3+b\,du^3=0, \tag{39}$$

then we have

$$a'r^2\lambda^3+b=0,$$

and the point of contact η coincides with the ray point $\sigma_{r\lambda}$. The family $F_{r\lambda}$ which forms with F_λ the net N_λ of the bundle B_r is then one of the three families of the Segre-Darboux system G for which the differential equation is

$$a'dv^3+b\,r\,du^3=0. \tag{40}$$

Therefore each family represented by equation (39) *can be associated with a family represented by equation* (40) *to form a net which belongs to the bundle* B_r, *—determined by the asymptotic net and a constant cross ratio* r,*—and which is such that the ray of* P_y *with respect to this net touches the envelope* (35) *in the ray point of* P_y *with respect to the component family given by equation* (40).

Similarly, if the family F_λ is any one of the three families for which the differential equation is

$$a'r\,dv^3+b\,du^3=0, \tag{41}$$

then the point of contact η coincides with the ray point σ_λ. The family $F_{r\lambda}$ is then one of the families for which the differential equation is

$$a'dv^3+b\,r^2\,du^3=0. \tag{42}$$

Therefore each family represented by equation (41) *can be associated with a family represented by equation* (42) *to form a net which belongs to the bundle* B_r *and is such that the ray of* P_y *with respect to this net touches the envelope* (35) *in the ray point of* P_y *with respect to the component family given by equation* (41).

VII. Bundles with conjugate nets fundamental

The bundle of nets determined on a surface S_y by a conjugate net and a constant cross ratio r is analogous to the bundle of conjugate nets, the asymptotic net of the latter being replaced by an arbitrary conjugate net, and -1 being replaced by r. In studying such a bundle we shall suppose

that the fundamental conjugate net is parametric. Then S_y is an integrating surface of a system of differential equations of the form*

$$\begin{aligned} y_{uu} &= a\ y_{vv}+b\ y_u+c\ y_v+d\ y, \\ y_{uv} &= \quad\ b'y_u+c'y_v+d'y. \end{aligned} \tag{43}$$

The differential equation of the asymptotic net is

$$a\ du^2+dv^2=0, \tag{44}$$

and the family conjugate to the family F_λ, represented by (5) in *conjugate* parameters, is given by the equation

$$a\ du+\lambda\ dv=0. \tag{45}$$

In order to find the ray point of P_y with respect to the family F_λ, let us observe that the point $y_u+\lambda y_v$ is on the tangent at P_y of the curve of F_λ through P_y. Any point Y, except P_y, on this tangent is defined by the formula

$$Y=y_u+\lambda y_v+Ky,$$

wherein K is a function of u, v which is to be determined so that the point Y shall be the required ray point. We find, on differentiating Y and making use of system (43),

$$\begin{aligned} Y_u &= ay_{vv}+(b+\lambda b'+K)y_u+(c+\lambda c'+\lambda_u)y_v+(d+\lambda d'+K_u)y, \\ Y_v &= \lambda y_{vv}+b'y_u+(c'+K)y_v+(d'+\lambda_v+K_v)y. \end{aligned}$$

The expression λY_u-aY_v does not contain y_{vv}, and is a linear combination of y and $y_u+\lambda y_v$ if, and only if, K satisfies the condition

$$(\lambda^2+a)K=-b'\lambda^3+(c'-b)\lambda^2+(c+ab')\lambda-ac'+\lambda\lambda_u-a\lambda_v.$$

Therefore the ray point σ_λ of P_y with respect to the family F_λ is given by the formula

$$\sigma_\lambda=(\lambda^2+a)\rho+\lambda(\lambda^2+a)\sigma+\lambda(Q\lambda+aP)y, \tag{46}$$

wherein we have placed

$$P=\frac{c}{a}+2b'-\frac{\lambda_v}{\lambda},\quad Q=2c'-b+\frac{\lambda_u}{\lambda}, \tag{47}$$

and ρ, σ are the covarients whose definitions are

$$\rho=y_u-c'y,\ \sigma=y_v-b'y. \tag{48}$$

* Green, *Projective differential geometry of one-parameter families of space curves*, American Journal of Mathematics, vol. 37 (1915) and vol. 38 (1916).

In fact, the points ρ and σ are the ray points of P_v with respect to the families $v=$ const. and $u=$ const. respectively.

If we replace λ by λh in equation (46), we obtain the ray point $\sigma_{\lambda h}$ of P_v with respect to the first family of the net $N_{\lambda h}$ represented by equation (9) in *conjugate* parameters. And if we use the triangle y, ρ, σ as a local triangle of reference, with suitably chosen unit point, the local coördinates of $\sigma_{\lambda h}$ are given by the equations

$$x_1=\lambda h(Q\lambda h+aP), \quad x_2=\lambda^2h^2+a, \quad x_3=\lambda h(\lambda^2h^2+a). \tag{49}$$

Eliminating h from these equations, and making the result homogeneous, we obtain

$$(ax_1^2+x_3^2)\,(4x_1-Qx_2-Px_3)+aQx_2^3-3aPx_2^2x_3-3Qx_2x_3^2+Px_3^3=0. \tag{50}$$

This is the ray-point cubic of P_v with respect to the system G determined by the parametric conjugate net and the family F_λ. It has a node at P_v, the double point tangents being the asymptotic tangents,

$$ax_2^2+x_3^2=0.$$

The cubic, moreover, has three inflexions which lie on the flex ray

$$4x_1-Qx_2-Px_3=0,$$

and each inflexion lies on one of the lines through P_v given by

$$aQx_2^3-3aPx_2^2x_3-3Qx_2x_3^2+Px_3^3=0.$$

It is interesting to compare the cubic (50) with the cubic (18). For this purpose we shall need to have the equations of the two curves referred to the same coördinate system. Let (x_1, x_2, x_3) be the coördinates of a point referred to the triangle used in deriving equation (18), and let $(\bar{x}_1, \bar{x}_2, \bar{x}_3)$ be the coördinates of the same point referred to the triangle used in the present section. Moreover, let the parametric conjugate net for system (43) be the conjugate net (11). Then the transformation of coördinates between these two systems is found to be given by the equations

$$\begin{aligned} x_1&=2\lambda\varphi_v\psi_v(\bar{x}_1-\mathfrak{E}'\bar{x}_2-\mathfrak{B}'\bar{x}_3),\\ x_2&=\quad -\psi_v\bar{x}_2+\varphi_v\bar{x}_3,\\ x_3&=\quad \lambda(\psi_v\bar{x}_2+\varphi_v\bar{x}_3), \end{aligned} \tag{51}$$

where λ is the function of the asymptotic parameters that appears in equation (11), and φ, ψ are solutions of the equations*

$$\varphi_u+\lambda\varphi_v=0, \quad \psi_u-\lambda\psi_v=0,$$

* Lane, loc. cit., p. 285.

while $\mathfrak{B}'$, $\mathfrak{C}'$ are the invariants usually defined by the formulas

$$(52) \qquad \mathfrak{B}'=\frac{1}{8}\left(4b'+2\frac{c}{a}-\frac{a_v}{a}\right), \quad \mathfrak{C}'=\frac{1}{8}\left(4c'-2b+\frac{a_u}{a}\right).$$

By means of the transformation (51) and the relation*

$$a\varphi_v^2+\psi_v^2=0,$$

equation (18) is transformed into

$$(53) \quad (ax_2^2+x_3^2)(x_1-\mathfrak{C}'x_2-\mathfrak{B}'x_3)+a\mathfrak{C}'x_2^3-3a\mathfrak{B}'x_2^2x_3-3\mathfrak{C}'x_2x_3^2+\mathfrak{B}'x_3^3=0,$$

the dashes being dropped from the result. Comparison of equations (50) and (53) shows that the two curves are the same if, and only if,

$$P=4\mathfrak{B}', \quad Q=4\mathfrak{C}'.$$

These conditions may be reduced, by means of the definitions (47) and (52), to the differential equations

$$\frac{\lambda_u}{\lambda}=\frac{1}{2}\frac{a_u}{a}, \quad \frac{\lambda_v}{\lambda}=\frac{1}{2}\frac{a_v}{a},$$

of which the most general solution is

$$\lambda=\text{const.}\sqrt{a}.$$

Therefore the ray-point cubics (50) *and* (18) *are the same if, and only if, the system G associated with the cubic* (50) *contains the asymptotic net.*

The transformation (51) is useful for many purposes. For example, the equation $x_1=0$ of the flex ray of the cubic (18) becomes, in the notation of conjugate nets,

$$x_1-\mathfrak{C}'x_2-\mathfrak{B}'x_3=0.$$

The equation of the Darboux tangents is, in this notation,

$$a\mathfrak{C}'x_2^3-3a\mathfrak{B}'x_2^2x_3-3\mathfrak{C}'x_2x_3^2+\mathfrak{B}'x_3^3=0,$$

and the equation of the Segre tangents is

$$a^2\mathfrak{B}'x_2^3+3a\mathfrak{C}'x_2^2x_3-3a\mathfrak{B}'x_2x_3^2-\mathfrak{C}'x_3^3=0.$$

Finally, the equation (22) of the ray conic becomes

$$(a\mathfrak{B}'^2+\mathfrak{C}'^2)(ax_2^2+x_3^2)=a(x_1-\mathfrak{C}'x_2-\mathfrak{B}'x_3^2)^2.$$

We shall now consider briefly the envelope which is analogous to the curve (35). If in equations (49) we replace λ by $r\lambda$, we obtain the coördinates

* Lane, loc. cit., p. 286.

of the ray point of P_y with respect to the second family of the net $N_{\lambda h}$. The line coördinates of the ray of P_y with respect to the net $N_{\lambda h}$ are then found to be given by the following formulas:

$$
\begin{aligned}
u_1 &= r^2\lambda^4h^4+a(1+r^2)\lambda^2h^2+a^2,\\
u_2 &= r\lambda^2h^2[-Qr\lambda^2h^2-aP(1+r)\lambda h+aQ],\\
u_3 &= a[Pr\lambda^2h^2-Q(1+r)\lambda h-aP].
\end{aligned}
\tag{54}
$$

The envelope of the ray, as $N_{\lambda h}$ varies over all the nets of the pencil determined by the net N_λ in the bundle B_r, is therefore ordinarily a curve of class four. If, however, $r=-1$, and if $\lambda^2h^2+a\neq 0$, equations (54) reduce to

$$u_1=\lambda^2h^2+a, \quad u_2=-Q\lambda^2h^2, \quad u_3=-aP,$$

so that the result of eliminating h is the linear relation

$$PQu_1+Pu_2+Qu_3=0.$$

Therefore we have the following conclusion. *If we consider a bundle of nets consisting of all nets every one of which has the property that at every surface point its tangents separate the tangents of a fundamental conjugate net harmonically, and if we consider a pencil of nets in this bundle, determined by a net N_λ of the bundle, then the rays of a surface point P_y with respect to all the nets of this pencil form a flat pencil of lines. If the fundamental conjugate net of the bundle is the parametric net of system* (43), *the center of the bundle is the point* (PQ, P, Q).

VIII. Bundles with non-conjugate non-asymptotic nets fundamental

In a paper published* posthumously, G. M. Green employed the differential equations of a surface S, referred to a non-conjugate non-asymptotic net, in the form

$$
\begin{aligned}
y_{uu} &= a\,y_{uv}+b\,y_u+c\,y_v+dy,\\
y_{vv} &= a'y_{uv}+b'y_u+c'y_v+d'y.
\end{aligned}
\tag{55}
$$

The differential equation of the asymptotic net in this notation is

$$a\,du^2+2\,du\,dv+a'dv^2=0, \tag{56}$$

and the family conjugate to a family F_λ, in these parameters, is given by

$$(a+\lambda)du+(a'\lambda+1)dv=0. \tag{57}$$

---

* Green, *Nets of space curves*, these Transactions, vol. 21 (1920), pp. 207-236.

If the parametric tangents form with the asymptotic tangents at every point of S a constant cross ratio r, then equation (4) shows that we must have

$$4r=(1+r)^2aa'. \tag{58}$$

Therefore the parametric net on an integrating surface of system (55) *belongs to the bundle determined by the asymptotic net and a constant cross ratio* $r\neq -1$ *if equation* (58) *is satisfied.*

In studying the bundle determined by an arbitrary net not conjugate and not asymptotic, and a constant cross ratio r, we take the arbitrary net as the parametric net of system (55). Then, by a course of reasoning similar to that employed in the preceding section, we find that the ray point σ_λ of a surface point P_y with respect to the family F_λ, in *arbitrary* parameters, is given by the formula

$$\sigma_\lambda=(\lambda^2a'+2\lambda+a)\rho+\lambda(\lambda^2a'+2\lambda+a)\sigma+\lambda(Q\lambda+P)y, \tag{59}$$

wherein we have placed

$$\begin{aligned} P&=ca'-c'a-b-a\frac{b'}{a'}-2\frac{c}{a}+\frac{\lambda_u}{\lambda}-a\frac{\lambda_v}{\lambda},\\ Q&=b'a-ba'-c'-a'\frac{c}{a}-2\frac{b'}{a'}-\frac{\lambda_v}{\lambda}+a'\frac{\lambda_u}{\lambda}, \end{aligned} \tag{60}$$

and ρ, σ are the covariants defined by the formulas

$$\rho=y_u+\frac{c}{a}y,\quad \sigma=y_v+\frac{b'}{a'}y. \tag{61}$$

The points ρ and σ are the ray points of P_y with respect to the curves $v=$const. and the curves $u=$const. respectively.

The ray point $\sigma_{\lambda h}$ of P_y with respect to the first family of the net $N_{\lambda h}$ has coördinates which, referred to the local triangle y, ρ, σ with suitably chosen unit point, are given by the following formulas:

$$\begin{aligned} x_1&=\lambda h(Q\lambda h+P),\\ x_2&=\lambda^2h^2a'+2\lambda h+a,\\ x_3&=\lambda h(\lambda^2h^2a'+2\lambda h+a). \end{aligned} \tag{62}$$

Eliminating h from these equations, we obtain

$$\begin{aligned} &(ax_2^2+2x_2x_3+a'x_3^2)[4(1-aa')x_1-(2P-aQ)x_2-(2Q-a'P)x_3]\\ &\quad+a(2P-aQ)x_2^3+3aa'Px_2^2x_3+3aa'Qx_2x_3^2+a'(2Q-a'P)x_3^3=0. \end{aligned} \tag{63}$$

This is the ray-point cubic of P_v with respect to the system G determined by the parametric net of system (55) and the family F_λ in arbitrary parameters. It has a node at P_v, the double point tangents being the asymptotic tangents. It also has three inflexions which lie on the flex ray

$$4(1-aa')x_1-(2P-aQ)x_2-(2Q-a'P)x_3=0,$$

and each inflexion lies on one of the lines through P_v,

$$a(2P-aQ)x_2^3+3aa'Px_2^2x_3+3aa'Qx_2x_3^2+a'(2Q-a'P)x_3^3=0.$$

Let us suppose that the asymptotic net belongs to the system G associated with the cubic (63). Then the cross ratio of the asymptotic tangents and the parametric tangents at every surface point is constant, so that $aa'=\text{const.}$ Moreover, the cross ratio of the parametric tangents, the tangent belonging to the family F_λ, and the tangent of either family of asymptotic curves, at every surface point, is constant, so that

$$\lambda=\text{const.}\frac{-1+\sqrt{1-aa'}}{a'}=\frac{\text{const.}}{a'}.$$

Then we find

$$2P-aQ=4a\mathfrak{C}'^{(u)},\quad 2Q-a'P=4a'\mathfrak{C}'^{(v)}, \tag{64}$$

where $\mathfrak{C}'^{(u)}$ and $\mathfrak{C}'^{(v)}$ are the invariants* whose vanishing implies that the curves C_u and C_v respectively are Darboux curves. Under these conditions, if $aa'\neq 4$, equation (63) becomes

$$\begin{aligned}(ax_2^2+2x_2x_3+a'x_3^2)&[(1-aa')x_1-a\mathfrak{C}'^{(u)}x_2-a'\mathfrak{C}'^{(v)}x_3]\\ &+a^2\mathfrak{C}'^{(u)}x_2^3-\frac{3a^2a'}{aa'-4}(2\mathfrak{C}'^{(u)}+a'\mathfrak{C}'^{(v)})x_2^2x_3\\ &-\frac{3aa'^2}{aa'-4}(a\mathfrak{C}'^{(u)}+2\mathfrak{C}'^{(v)})x_1x_3^2+a'^2\mathfrak{C}'^{(v)}x_3^3=0.\end{aligned} \tag{65}$$

This is the ray-point cubic of the system G which is determined by the asymptotic net and one family of the parametric curves of system (55), *and which also contains the other parametric family.* If $aa'=4$, equation (58) shows that the cross ratio of the parametric tangents and asymptotic tangents at every surface point is a root of the equation $1+r+r^2=0$.

---

* Green, *Nets of space curves*, p. 217.

The line coördinates of the ray of P_v with respect to the net $N_{\lambda h}$ represented by equation (9) in arbitrary parameters are found to be given by the following equations, analogous to equations (54):

$$
\begin{aligned}
u_1 &= r^2a'^2\lambda^4h^4+2a'r(1+r)\lambda^3h^3+(4r+aa'+aa'r^2)\lambda^2h^2+2a(1+r)\lambda h+a^2,\\
u_2 &= -r\lambda^2h^2\left[a'Qr\lambda^2h^2+a'P(1+r)\lambda h+2P-aQ\right], \qquad (66)\\
u_3 &= -(2Q-a'P)r\lambda^2h^2-aQ(1+r)\lambda h-aP,
\end{aligned}
$$

where P and Q are the functions defined in equations (60). *The envelope of this ray as the net $N_{\lambda h}$ varies over all the nets of the pencil determined by the net N_λ in the bundle B_r is, therefore, ordinarily of class four. However, if $r=-1$, equations* (66) *reduce to*

$$
\begin{aligned}
u_1 &= a'^2\lambda^4h^4+2(aa'-2)\lambda^2h^2+a^2,\\
u_2 &= \lambda^2h^2(-a'Q\lambda^2h^2+2P-aQ), \qquad (67)\\
u_3 &= (2Q-a'P)\lambda^2h^2-aP,
\end{aligned}
$$

and the envelope is a conic.

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ON THE MOMENTAL CONSTANTS OF A SUMMABLE FUNCTION*

BY

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I. Introduction

The definition of the momental constants ν_n of a summable function $f(x)$, and the first deductions concerning their relations with the generating function f were given by Haskins in 1916.† Proofs were supplied by both Jackson‡ and Van Vleck§ for the fundamental theorem enunciated by Haskins concerning these constants. Van Vleck, moreover, by his definition and use of the "measure function" showed the existence of a relation between the momental constants of any one function and the moments of another associated with it.

The point of view maintained in the papers cited is roughly this: given a function $f(x)$ (or a pair of functions $f(x)$ and $\varphi(x)$), to determine certain of its (or their) characteristics by a study of the set (or sets) of momental constants. The problem dealt with in the present paper is on the other hand that of deducing a set of necessary and sufficient conditions that an enumerable set of constants be momental constants, the generating function to be defined on an arbitrarily chosen interval and to possess certain specified characteristics.

Section II in the following contains the deduction of certain necessary conditions. In Section III the sufficiency of these conditions is established. This end is attained by drawing upon Stieltjes' classic solution of the "problem of moments," and constructing theoretically through its use a function having the given constants as its momental constants. The function so derived is monotonic and as such is a typical representative of its class under Haskins' classification of summable functions. For the basis of this classification and for a statement of the significance of the momental constants the reader is referred to the original paper by Haskins.||

II. The deduction of certain necessary conditions that a set of constants be momental constants

1. **Introduction.** In this section a function $f(x)$ of specified character is considered and certain relations satisfied by its momental constants are

* Presented to the Society, February 28, 1925; received by the editors in January, 1925.

† These Transactions, vol. 17 (1916), p. 181.

‡ These Transactions, vol. 17 (1916), p. 178.

§ These Transactions, vol. 18 (1917), p. 326.

|| Loc. cit., pp. 185 and 194.

deduced. These relations are independent of the choice of $f(x)$, subject only to the enumerated specifications. Hence they constitute conditions which must necessarily be fulfilled by any set of constants if they are to be the momental constants of a function of the kind in question.

2. **Character of functions considered.** Suppose given any function $f(x)$ possessed of the following characteristics:

(A) it is a real, singled-valued function of the real variable x, defined at every point of an interval $a \leqq x \leqq b$;

(B) it is integrable in the sense of Lebesgue over the interval (a, b);

(C) it has on the interval (a, b) a finite upper measurable bound H* and a finite lower measurable bound h.

3. **The momental constants.** The hypotheses (B) and (C) imply the summability over (a, b) of the functions $\{f(x)\}^n$ for all non-negative integral values of n. It is assured, therefore, that the constants

$$(1) \qquad \nu_n = \frac{1}{b-a}\int_a^b \{f(x)\}^n dx \qquad (n=0, 1, 2, \cdots),$$

exist. They are by definition the momental constants of $f(x)$ on (a, b), and it is to a deduction of relations satisfied by them that we proceed.

4. **A first necessary condition.** Setting $n=0$ in (1) we obtain the fact that

$$(a) \qquad \nu_0 = 1 .$$

5. **A second necessary condition.** From hypotheses (A) and (C) it follows that the function $\{f(x)\}^2$ is non-negative and is possessed on (a, b) of a finite upper measurable bound. As such it fulfils the hypotheses of a theorem by Haskins† which asserts for the case in hand that if μ_n is the nth momental constant of $\{f(x)\}^2$ and if L^2 is the upper measurable bound of $\{f(x)\}^2$ then $\lim_{n\to\infty} \mu_n/\mu_{n-1} = L^2$. Inasmuch as the substitution of $\{f(x)\}^2$ for $f(x)$ in (1) is equivalent to the substitution of $2n$ for n, it is clear that $\mu_n = \nu_{2n}$. We have, therefore, as a second result

$$(b) \qquad \lim_{n\to\infty} \frac{\nu_{2n}}{\nu_{2n-2}} = L^2.$$

The constant L^2 is readily seen to be identical with the larger of the two numbers H^2 and h^2.

* For the definition of the measurable bounds see Haskins, loc. cit., p. 184.

† Loc. cit., p. 188.

6. **Deduction of a relation between the momental constants and Stieltjes' moments.** For use in the further deductions let the function $F(x)$ be defined by the identity

$$f(x)+L \equiv F(x) . \tag{2}$$

Since $L \geqq |h|$, the lower measurable bound of $F(x)$ on (a, b) is positive or zero. It is with the purpose of introducing such a function to replace $f(x)$ in our considerations that the substitution (2) is made.

By virtue of relations (2) and (1) we have now

$$\nu_n = \frac{1}{b-a}\int_a^b \{F(x)-L\}^n dx ,$$

or, upon expanding the integrand,

$$\nu_n = \frac{1}{b-a}\sum_{j=0}^{n}\binom{n}{j}(-L)^j \int_a^b \{F(x)\}^{n-j} dx. \tag{3}$$

The Lebesgue integrals in this expression may be transformed into Stieltjes' integrals by the following method due to Van Vleck.*

The identical relation

$$\psi(y) \equiv mE(F(x) \leqq y)\dagger \tag{4}$$

defines a real single valued function of the real variable y which is known as the measure function of $F(x)$. It is seen to measure the distribution of the functional values of $F(x)$ over the range $-\infty < y < \infty$. Now

$$mE(F(x)<0)=0 , \text{ and } mE(F(x)>H+L)=0 .\ddagger$$

The structure of the Lebesgue integrals in question is, therefore, explicitly given by the formulas

$$\int_a^b \{F(x)\}^j dx \begin{cases} = \lim \sum_0^{H+L} mE(y_{i-1}<F(x)\leqq y_i)+mE(F(x)=0) , \text{ for } j=0 , \\ = \lim \sum_0^{H+L} y_i^j mE(y_{i-1}<F(x)\leqq y_i) , \text{ for } j=1 , 2 , \cdots . \end{cases}$$

---

* Loc. cit., p. 327.

† The difference between this formula and Van Vleck's is accounted for by the fact that he considers only bounded functions while here $F(x)$ is not so restricted.

‡ From the definition of the measurable bounds.

Since the element under the sign of summation in terms of the function $\psi(y)$ is precisely

$$y_i^j\{\psi(y_i)-\psi(y_{i-1})\}=y_i^j\Delta_i\psi(y)$$

we have the identities

$$(5)\quad \int_a^b\{F(x)\}^j dx \begin{cases} =\int_0^{H+L} d\psi(y)+\psi(0)\ , \text{ for } j=0, \\ =\int_0^{H+L} y^j d\psi(y)\ , \text{ for } j=1, 2, \cdots . \end{cases}$$

From (4), $\psi(y)$ is seen to be a monotonic non-decreasing function. It may be considered, therefore, to define a distribution of mass over the positive axis of y, the value $\psi(y)$ representing for any y the mass on the interval $(0, y)$. Denoting by c_j the jth moment of this distribution about $y=0$ we have, since $\psi(y)$ is constant for $y>H+L$,

$$(6)\quad c_j=\begin{cases}\psi(H+L), & \text{for } j=0, \\ \int_0^{H+L} y^j d\psi(y), & \text{for } j=1, 2, \cdots . \end{cases}$$

The substitution of (6) into (5) and (3) produces the relation

$$(7)\quad \nu_n=\frac{1}{b-a}\sum_{j=0}^{n}\binom{n}{j}(-L)^j c_{n-j}\ .$$

On the other hand the substitution of (2) into (5) and the expansion of the result yields in virtue of (6) the inverse relation, namely

$$(8)\quad c_n=(b-a)\sum_{j=0}^{n}\binom{n}{j}L^j\nu_{n-j}\ .$$

By means of formula (8) any relation subsisting between the constants c_n may be translated into a relation fulfilled by the constants ν_n. The following deduction of such a relation was given (except for small modifications) by Stieltjes.*

7. **Relations satisfied by Stieltjes' moments.** The quadratic form

$$F_{p,m}=\int_0^{H+L} y^p\{T_0+yT_1+\cdots+y^mT_m\}^2 d\psi(y)+\delta_{0,p}\psi(0)T_0^2\ ,$$

* Annales de la Faculté des Sciences de Toulouse, vol. 8 (1894), p. J21, or *Œuvres Complètes*, vol. 2, p. 422.

in which $\delta_{0,p}=0$ for $p\neq 0$ and $\delta_{0,0}=1$, is expressible alternatively in the forms

$$F_{p,m}=\int_0^{H+L} y^p \sum_{i,j=0}^{m} T_iT_j y^{i+j}\,d\psi(y)+\delta_{0,p}\psi(0)T_0^2$$

$$=\sum_{i,j=0}^{m} T_iT_j\int_0^{H+L} y^{p+i+j}\,d\psi(y)+\delta_{0,p}\psi(0)T_0^2,$$

or by virtue of (6) in the form

$$F_{p,m}=\sum_{i,j=0}^{m} c_{p+i+j}T_iT_j.$$

From the first expression for $F_{p,m}$ above it is clear, however, that the form is positive definite. Consequently its determinant is positive, and reading this determinant from the last expression given we have

$$D_{p,m}=\begin{vmatrix} c_p & c_{p+1} & \cdot & \cdot & \cdot & c_{p+m} \\ c_{p+1} & c_{p+2} & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ c_{p+m} & \cdot & \cdot & \cdot & \cdot & c_{p+2m} \end{vmatrix}>0\,. \tag{9}$$

This result obviously holds for p and m any positive integers or zero. It holds then in particular for $p=0$ and $p=1$.*

8. **Further necessary conditions on the ν_n.** The substitution from (8) in (9) yields a further relation among the momental constants ν_n. To facilitate the reduction of this relation we shall use in place of formula (8) an equivalent symbolic form obtained as follows. Comparing the sum in (8) with the polynomial $(u+L)^n$ it is seen that the former may be obtained from the latter by replacing each power of u by the constant ν with the corresponding subscript, i. e. by replacing u^i by ν_i, $i=0, 1, 2, \cdots$. If it is agreed to indicate that this change is to be made in a given polynomial by enclosing in brackets, [ ], we may write (8) in the form

$$c_n=(b-a)[(u+L)^n]. \tag{10}$$

---

* The converse, namely, if $D_{0,m}$ and $D_{1,m}$ are positive for $m=0, 1, 2, \cdots$ then $D_{p,m}$ is positive for all non-negative integral values of p and m, is not required in the reasoning of this paper. It can be shown, however, by the use of the determinant formula (8) page 26 of H. Weber's *Lehrbuch der Algebra*, Klein Ausgabe, Braunschweig, 1921.

Substituting from (10) in the determinant of (9) for $p=0$ and removing the factor $(b-a)$ from each row we obtain the form

$$(11)\qquad D_{0,m}=(b-a)^{m+1}\begin{vmatrix} [1] & [(u+L)] & \cdots & [(u+L)^m] \\ [(u+L)] & [(u+L)^2] & \cdots & \cdot \\ [(u+L)^2] & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ [(u+L)^m] & \cdot & \cdots & [(u+L)^{2m}] \end{vmatrix}.$$

This may be simplified by replacing the jth column successively for $j=m+1, m, \cdots, 2$, by the combination [column j − column $(j-1)$]. Because of the relation

$$[u^s(u+L)^r]-L[u^s(u+L)^{r-1}]=[u^{s+1}(u+L)^{r-1}],$$

which is easily verified, the net result of the operation is to replace in each element of the columns operated on one factor $(u+L)$ within brackets by a factor u. A repetition of the operation in obvious fashion on both columns and rows ultimately results in replacing all factors $(u+L)$ in this way. Dropping the symbolism and setting

$$A_m=D_{0,m}/(b-a)^{m+1}$$

we have the result

$$A_m=\begin{vmatrix} \nu_0 & \nu_1 & \nu_2 & \cdots & \nu_m \\ \nu_1 & \nu_2 & \cdot & \cdots & \cdot \\ \nu_2 & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \nu_m & \cdot & \cdot & \cdots & \nu_{2m} \end{vmatrix}.$$

Again, substituting from (10) in the determinant of (9) for $p=1$ we obtain the form

$$(12)\qquad D_{1,m}=(b-a)^{m+1}\begin{vmatrix} [(u+L)] & [(u+L)^2] & \cdots & [(u+L)^{m+1}] \\ [(u+L)^2] & [(u+L)^3] & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ [(u+L)^{m+1}] & \cdot & \cdots & [(u+L)^{2m+1}] \end{vmatrix}.$$

The same sequence of operations as was resorted to in the preceding case reduces this to the form

$$D_{1,m}=(b-a)^{m+1}\begin{vmatrix} [(u+L)] & [u(u+L)] & \cdots & [u^m(u+L)] \\ [u(u+L)] & [u^2(u+L)] & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ [u^m(u+L)] & \cdot & \cdots & [u^{2m}(u+L)] \end{vmatrix}.$$

A further reduction is attained by replacing this $(m+1)$-rowed determinant by an equivalent one of $(m+2)$ rows, namely by

$$D_{1,m}=(b-a)^{m+1}\begin{vmatrix} 1 & 0 & 0 & & 0 \\ [1] & [(u+L)] & [u(u+L)] & \cdots & [u^m(u+L)] \\ [u] & [u(u+L)] & \cdot & \cdots & \cdot \\ [u^2] & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ [u^m] & [u^m(u+L)] & \cdot & \cdots & [u^{2m}(u+L)] \end{vmatrix}.$$

Performing now once more on the columns successively for $j=2, 3, \cdots, (m+2)$, the same operation as was previously employed results in replacing the remaining factors $(u+L)$ by u. Dropping the symbolism and setting

$$B_m=D_{1,m}/(b-a)^{m+1}$$

we have

$$B_m=\begin{vmatrix} 1 & (-L) & (-L)^2 & \cdots & (-L)^{m+1} \\ \nu_0 & \nu_1 & \nu_2 & \cdots & \nu_{m+1} \\ \nu_1 & \nu_2 & \cdot & \cdots & \cdot \\ \nu_2 & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \nu_m & \cdot & \cdot & \cdots & \nu_{2m+1} \end{vmatrix}.$$

Since A_m and B_m differ from the positive determinants $D_{0,m}$ and $D_{1,m}$ respectively only by the positive factor $(b-a)^{m+1}$ we have as a third relation

(c) $$A_m>0, \quad B_m>0 \qquad (m=0, 1, 2, \cdots).$$

Relations (a), (b), and (c) together comprise a set of necessary conditions on any set of constants ν_n if they are to be the momental constants of a function possessed of properties (A), (B), and (C).

III. Proof of the sufficiency of the set of conditions deduced

9. **Introduction.** In this section an enumerably infinite set of constants ν_n satisfying the relations (a), (b), and (c) will be assumed as given, together with an interval $a \leqq x \leqq b$ for the real variable x. A function $f(x)$ defined on (a, b) and possessing the ν_n as its momental constants will be constructed, and it will be shown that this function is possessed of the characteristics (A), (B), and (C). With the success of this procedure the sufficiency of the conditions imposed upon the constants ν_n is established.

10. **Hypotheses concerning the constants ν_n.** Suppose given the set of constants ν_n, satisfying the relations

(a) $$\nu_0 = 1,$$

(b) $$\lim_{n \to \infty} \frac{\nu_{2n}}{\nu_{2n-2}} = L^2,$$

(c) $$A_m = \begin{vmatrix} \nu_0 & \nu_1 & \cdots & \nu_m \\ \nu_1 & \nu_2 & \cdots & \\ \nu_2 & \cdots & & \\ \cdots & & & \\ \cdots & & & \\ \nu_m & \cdots & & \nu_{2m} \end{vmatrix} > 0, \quad B_m = \begin{vmatrix} 1 & (-L) & \cdots & (-L)^{m+1} \\ \nu_0 & \nu_1 & \cdots & \nu_{m+1} \\ \nu_1 & \nu_2 & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \nu_m & \cdots & \cdots & \nu_{2m+1} \end{vmatrix} > 0$$

$$(m = 0, 1, 2, \cdots).$$

11. **Introduction of a related set of constants.** Consider first the conditions (c). The reverse of the operations through which the forms A_m and B_m were deduced in the preceding section show these determinants to differ only by a positive factor respectively from the determinants $D_{0,m}$ and $D_{1,m}$ as given by formulas (11) and (12). Let the new set of constants c_n $(n = 0, 1, 2, \cdots)$ be defined now by means of formula (8), namely

$$c_n = (b-a) \sum_{j=0}^{n} \binom{n}{j} L^j \nu_{n-j},$$

or its equivalent (10),

$$c_n = (b-a)[(u+L)^n].$$

The steps following the derivation of the relation (9) in the preceding chapter are now reversible. Retracing them, the conditions (c) on the set of constants ν_n are found to be equivalent to the conditions

$$(13)\qquad \begin{vmatrix} c_0 & c_1 & \cdots & c_m \\ c_1 & c_2 & \cdots & \cdot \\ c_2 & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ c_m & \cdot & \cdots & c_{2m} \end{vmatrix} > 0, \qquad \begin{vmatrix} c_1 & c_2 & \cdots & c_{m+1} \\ c_2 & c_3 & \cdots & \cdot \\ c_3 & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ c_{m+1} & \cdot & \cdots & c_{2m+1} \end{vmatrix} > 0,$$

on the constants of the associated set c_n.

12. **Outline of Stieltjes' solution of the problem of moments.** The conditions (13) on a set of constants are familiar ones. Stieltjes showed* that when they are fulfilled then the series

$$\frac{c_0}{z} - \frac{c_1}{z^2} + \frac{c_2}{z^3} - \cdots$$

is the formal expansion of a continued fraction of the type

$$\frac{1}{a_1 z} + \frac{1}{a_2} + \frac{1}{a_3 z} + \cdots,$$

with positive coefficients a_i. On the other hand he showed how the convergents of such a fraction may be utilized in the construction o a sequence of functions $\varphi_n(y)$ from which an associated function $\Phi(y)$ may be determined. The existence of this function solves the "problem of moments," for it follows from its mode of derivation first that $\Phi(y)$ defines a distribution of mass in the manner of $\psi(y)$ in the preceding section and second that

$$\int_0^\infty y^n d\Phi(y) = c_n \qquad (n = 0, 1, 2, \cdots).$$

Two cases are distinguished by Stieltjes in his deductions. Under the conditions (13) the ratio c_{n+1}/c_n increases with n.† He showed that when it approaches a finite limit the function $\Phi(y)$ is determined uniquely save for its values at its points of discontinuity. In the alternative case there are infinitely many such functions.

* Loc. cit., Annales, p. J1, *Œuvres* p. 402.

† This can be shown as follows: By the footnote on page 172, conditions (13) imply that $D_{n,0} > 0$, $D_{n+1,0} > 0$, and $D_{n,1} > 0$ for all positive n. Hence $D_{n,1}/(D_{n,0}D_{n+1,0}) > 0$, that is, $c_{n+2}/c_{n+1} > c_{n+1}/c_n$.

We shall not distinguish at this time between the two cases, but shall summarize those of Stieltjes' results which are of importance for the purpose before us as follows. Given any set of constants c_n satisfying the conditions (13), there exists at least one function $\Phi(y)$ which

(i) is defined and single valued for $0 \leqq y < \infty$,

(ii) is monotonic non-decreasing,

(iii) vanishes at zero, i. e. $\Phi(0)=0$,

(iv) satisfies the relations

$$\int_0^\infty y^n d\Phi(y) = c_n \qquad (n=0, 1, 2, \cdots).$$

13. **The function $\psi(y)$.** The properties enumerated restrict the discontinuities of $\Phi(y)$ to occur in the points of an enumerable set.* Because of this the values taken on by $\Phi(y)$ at its points of discontinuity (excepting the point $y=0$ if it is among them) may be changed at pleasure without affecting the integrals in (iv). In particular the value at every such point may be made equal to the right hand limiting value, i. e.

$$\Phi(y) = \lim_{\substack{\epsilon \to 0 \\ \epsilon > 0}} \Phi(y+\epsilon) .$$

We shall suppose this to have been done, and since this alteration is to be made at $y=0$ also, if $\Phi(y)$ is discontinuous there we shall denote the new function so obtained by $\psi(y)$. Obviously $\psi(y)$ is possessed of properties (i) and (ii). In place of (iii) and (iv) it has the characteristics that

(iii)′ it is non-negative, i. e. $\psi(0) \geqq 0$,

(iv)′ it satisfies the relations

$$\int_0^\infty d\psi(y) + \psi(0) = c_0 ,$$

$$\int_0^\infty y^n d\psi(y) = c_n \qquad (n=1, 2, \cdots).$$

In addition we have by construction that

(v) it is continuous toward the right.

14. **A lemma.** The end toward which our deductions point is the construction of a suitable function of x. The following lemma is to serve as a basis upon which such a function may be derived from the $\psi(y)$ at hand.

* See Stieltjes, loc. cit., chap. vi.

LEMMA. *Given a function $\omega(y)$ which is*
(α) *defined and single-valued for $0 \leqq y < \infty$ or $0 \leqq y \leqq Y$,*
(β) *monotonic non-decreasing,*
(γ) *non-negative on its interval of definition,*
(δ) *continuous to the right, i. e.*

$$\omega(y) = \lim_{\substack{\epsilon \to 0 \\ \epsilon > 0}} \omega(y+\epsilon) .$$

Then if $M(x)$ is the measure function of $\omega(y)$, conversely $\omega(y)$ is the measure function of $M(x)$.

The function $\omega(y)$ has 0 as a lower bound, whence, by the definition of the measure function for the case in hand,

$$M(x) \equiv mE_y(0 \leqq \omega(y) \leqq x) . \tag{14}$$

The subscript (y in this case) is appended to the E to assist one in following the reasoning. It denotes the axis along which the point set in question is measured. It is clear from the identity (14) that every measure function is possessed of the properties (α), (β), (γ), and (δ) as enumerated above. If we denote the measure function of $M(x)$ by $\Omega(y)$ we have accordingly

$$\Omega(y) \equiv mE_x(0 \leqq M(x) \leqq y) . \tag{15}$$

Moreover, $\Omega(y)$ will have the properties (α), (β), (γ), and (δ) in common with $\omega(y)$. We wish to show that $\Omega(y) \equiv \omega(y)$. Because of the right-handed continuity of both functions it will suffice to show that their values are the same at every point at which $\omega(y)$ is continuous. We shall consider separately the cases when y is a value at which $\omega(y)$ is increasing and when y is a value at which $\omega(y)$ is stationary.

Case 1. Let $y = y_1$ be a point at which (i) $\omega(y)$ is continuous, and (ii) $\omega(y)$ is increasing, i. e.,

$$\omega(y_1) < \omega(y_1+\epsilon) \text{ for every } \epsilon > 0 .$$

By hypothesis $\omega(y)$ is defined at every point of the interval $0 \leqq y \leqq y_1$. Moreover it is monotonic and non-negative. It follows that

$$mE_y(0 \leqq \omega(y) \leqq \omega(y_1)) = y_1 . \tag{16}$$

Defining x_1 by the relation $x_1 = \omega(y_1)$ and comparing (16) with (14) we see that

$$\text{if } x_1 = \omega(y_1), \quad \text{then} \quad M(x_1) = y_1. \tag{17}$$

Now since $\omega(y)$ is continuous and increasing at the point $y = y_1$, a small increase in y_1 is correlated with a small increase in $\omega(y_1)$, namely in x_1.

Because of the relation $M(x_1)=y_1$ from (17) this means that $M(x)$ is continuous and increasing at $x=x_1$.

Proceeding, one sees from (14) that $M(x)$ is defined at every point of the interval $0\leqq x\leqq x_1$. Inasmuch as it is also non-decreasing and non-negative the reasoning which leads from (14) to (17) may be applied again and leads from (15) to the fact

(18) $$\text{if } y_1=M(x_1) \quad \text{then} \quad \Omega(y_1)=x_1 .$$

In consequence of (17) and (18) we have

$$\Omega(y_1)=\omega(y_1) .$$

Case 2. Let y be a point in which $\omega(y)$ is continuous but stationary, i. e.

$$\omega(y)\equiv\omega(y+\epsilon) \quad \text{for} \quad 0\leqq\epsilon<\delta .$$

In this case $\omega(y)$ is constant throughout an interval, namely

$$\omega(y)\equiv x_1 \quad \text{for} \quad y_1\leqq y<y_2 .$$

From formula (14), then,

$$M(x_1-0)=mE_y(0\leqq\omega(y)<x_1)=y_1 ,$$

while

$$M(x_1)=mE_y(0\leqq\omega(y)\leqq x_1)=y_2 .$$

The function $M(x)$ is seen to be discontinuous in $x=x_1$. It takes on no values lying between y_1 and y_2 and hence

(19) $$mE_x(0\leqq M(x)\leqq y)\equiv mE_x(0\leqq M(x)\leqq y_1) ,$$

when $y_1\leqq y<y_2$. But since $M(x)$ is defined for every x on the interval $0\leqq x\leqq x_1$, and is non-decreasing and non-negative, the right hand member is simply x_1. The left hand member on the other hand is $\Omega(y)$ by (15). Hence (19) yields $\Omega(y)\equiv x_1$ for $y_1\leqq y<y_2$. This together with the hypothesis of Case 2 gives

$$\Omega(y)\equiv\omega(y) , \quad \text{for} \quad y_1\leqq y<y_2 .$$

This concludes the proof of the lemma.

15. **Application of the lemma.** Returning now to $\psi(y)$, let its measure function be constructed and denoted by $F(x)$. This function is defined on the interval $0\leqq x\leqq\psi(\infty)$. But the properties (i), (ii), (iii)′ and (v) of $\psi(y)$ are precisely the properties (α) to (δ) of the hypothesis for $\omega(y)$ in the lemma. It follows then by the lemma that $\psi(y)$ is in turn the measure function of

$F(x)$. By the reasoning employed to establish the relation (5) we may deduce then that

$$\int_0^{\psi(\infty)} \{F(x)\}^n dx \begin{cases} = \int_0^{\infty} d\psi(y) + \psi(0) \ , \ \text{for } n=0 \ , \\ = \int_0^{\infty} y^n d\psi(y) \ , \ \text{for } n=1, 2 \ , \ \cdots \ . \end{cases}$$

But because of the property (iv)′ of $\psi(y)$ the right hand expressions for any n are precisely c_n. Since (iv)′ yields also $\psi(\infty)=c_0$, it follows that the function $F(x)$ constructed in the manner shown satisfies the relations

$$\int_0^{c_0} \{F(x)\}^n dx = c_n \qquad (n=0, 1, 2 \ , \ \cdots) \ . \tag{20}$$

16. **Construction of the function $f(x)$ having the ν_n as its momental constants.** Formula (8) defines the constants c_n in terms of the constants ν_n. The inverse relation is given by (7), and substituting in this the value of c_n as given by (20) we obtain

$$\nu_n = \frac{1}{b-a} \sum_{j=0}^{n} \binom{n}{j} (-L)^j \int_0^{c_0} \{F(x)\}^{n-j} dx \ ,$$

that is,

$$\nu_n = \frac{1}{b-a} \int_0^{c_0} \{F(x)-L\}^n dx \ . \tag{21}$$

From (8) and hypothesis (a) we have, moreover, $c_0=(b-a)$, which substituted in (21) shows the interval of integration to be from 0 to $(b-a)$. This is changed to the given interval (a, b) by further setting

$$\begin{aligned} x' &= x+a \ , \\ F(x)-L &= f(x') \ . \end{aligned} \tag{22}$$

Upon dropping the primes on the variable of integration the resulting expression becomes

$$\nu_n = \frac{1}{b-a} \int_a^b \{f(x)\}^n dx \ .$$

The function $f(x)$ therefore has the given constants ν_n as its momental constants on (a, b). The characteristics of $f(x)$ remain to be determined.

17. **The properties of $f(x)$.** A review of the mode of derivation given readily shows that $f(x)$ is possessed of the characteristics (A) and (B)

enunciated at the beginning of this section. Further, we have by hypothesis (*b*) that

$$\lim_{n\to\infty} \frac{\nu_{2n}}{\nu_{2n-2}} = L^2 .$$

But ν_{2n} is the nth momental constant of the non-negative function $\{f(x)\}^2$. It follows by Haskins' theorem* that $\{f(x)\}^2$ has the upper measurable bound L^2, and hence that the measurable bounds of $f(x)$ are restricted from exceeding in absolute value the constant L. The function $f(x)$ is, therefore, also possessed of the property (*C*).

Summarizing our results we have the

THEOREM. *The conditions* (*a*), (*b*), *and* (*c*) *enunciated above are both necessary and sufficient that the enumerable set of constants* ν_n *be the momental constants of a function possessed of the properties* (*A*), (*B*), *and* (*C*).

IV. CONCLUDING REMARKS

18. **Introduction.** With the result of the preceding section the principal problem of this paper is solved. It is clear, however, that the procedure of Section III besides constituting a proof of the sufficiency of the conditions on the ν_n, embodies also a theoretical method for the construction of a non-decreasing function which is continuous on the right and has the ν_n as its momental constants. The uniqueness of this function may be shown as follows.

19. **The uniqueness of the solution.** The measurable bounds of $f(x)$ on (a, b) do not exceed in absolute value the constant L. Hence the measurable bounds of the function $F(x)$ in (18) are restricted to lie on the range from 0 to $2L$. This implies, however, that $\psi(y)$, the measure function of $F(x)$, is constant for $y \geqq 2L$, or in other words that the distribution of matter defined by it is confined to a finite interval. But it was shown by Van Vleck† that under these conditions the ratio c_{n+1}/c_n approaches a finite limit. This is the distinguishing characteristic of Stieltjes' determinate case, namely the case in which $\Phi(y)$ is unique save for its values in the points of discontinuity. The function $\psi(y)$ satisfying the conditions enumerated in Section III is, therefore, unique.

Suppose now that $f_2(x)$ is distinct from $f(x)$ but shares with it the properties that its momental constants are ν_n and that it is non-decreasing and continuous toward the right. Because of the last named property the points in which $f(x)$ and $f_2(x)$ are distinct comprise a set of positive measure.

* See note on page 169.

† Loc. cit., p. 330.

Hence the related functions $F(x)$ and $F_2(x)$ will also differ on the points of a set of positive measure and it follows that their measure functions $\psi(y)$ and $\psi_2(y)$ are not identical. Since they share the properties enumerated for $\psi(y)$ in Section III this is in contradiction with the uniqueness of $\psi(y)$ already shown. The hypothesis that $f_2(x)$ and $f(x)$ are distinct is, therefore, untenable.

If the requirement of right handed continuity is dropped the uniqueness of $f(x)$ at its points of discontinuity is lost, for an alteration of the values of $f(x)$ at these points need not impair its non-decreasing character and will not affect its Lebesgue integral. Otherwise, however, $f(x)$ remains determined and hence the deductions made constitute a new proof of the fact observed by Haskins and first proved by Jackson,* namely that the non-decreasing function with given momental constants is determined uniquely save at the points of an enumerable set.

20. **A means for obtaining the typical representative of a class of summable functions.** One further observation may be made, namely, if $\bar{f}(x)$ is a bounded summable function and belongs to a certain class C under Haskins' classification,† then $f(x)$, the measure function of the measure function of $\bar{f}(x)$, also belongs to C and is a typical representative of that class.

The proof of this is immediate. If H and h are the bounds of $\bar{f}(x)$, and $M(y)$ is its measure function on (a, b), then

$$\int_a^b \{\bar{f}(x)\}^n dx = \int_h^H y^n dM(y) + h^n M(h) .$$

By definition $f(x)$ is the measure function of $M(y)$, and since $M(y)$ satisfies the hypotheses of the lemma in Section III (a fact which is easily verified) it follows that $M(y)$ is in turn the measure function of $f(x)$. We have, therefore, also

$$\int_a^b \{f(x)\}^n dx = \int_h^H y^n dM(y) + h^n M(h) ,$$

that is,

$$\int_a^b \{f(x)\}^n dx \equiv \int_a^b \{\bar{f}(x)\}^n dx ,$$

i. e. $f(x)$ has the same momental constants as $\bar{f}(x)$, and so is a member of the class C. Being a measure function, however, $f(x)$ is non-decreasing. This completes the proof.

* These Transactions, vol. 17 (1916), p. 179.

† See last foot note on page 168.

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FUNDAMENTAL SYSTEMS OF FORMAL MODULAR PROTOMORPHS OF BINARY FORMS*

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I

If the binary form

$$(1) \qquad f(x\ ,\ y)=a_0x^q+\binom{q}{1}a_1x^{q-1}y+\binom{q}{2}a_2x^{q-2}y^2+\cdots+a_qy^q,$$

where $\binom{q}{1}$, $\binom{q}{2}$, $\cdots$ are the binomial coefficients and a_0, a_1, $\cdots$, a_q are independent variables, we make the substitution

$$(2) \qquad \begin{aligned} x&=X+tY\ ,\\ y&=Y\ , \end{aligned}$$

we obtain a binary form

$$(3) \qquad F(X,\ Y)=A_0X^q+\binom{q}{1}A_1X^{q-1}Y+\cdots+A_qY^q\ ,$$

in which

$$(4) \qquad A_j=a_0t^j+\binom{j}{1}a_1t^{j-1}+\cdots+a_j \qquad (j=1,\ 2,\ \cdots,\ q)\ .$$

A polynomial $P(a_0,\ a_1,\ \cdots,\ a_q)$ is said to be an algebraic *seminvariant* of $f(x,\ y)$ if, for all real or complex values of t,

$$P(A_0\ ,\ A_1\ ,\ \cdots\ ,\ A_q)=P(a_0\ ,\ a_1\ ,\ \cdots\ ,\ a_q)\ ;$$

a *formal modular seminvariant*, mod p, of $f(x,\ y)$ if, for all integral values of t,

$$P(A_0\ ,\ A_1\ ,\ A_2\ ,\ \cdots\ ,\ A_q)\equiv P(a_0\ ,\ a_1\ ,\ \cdots\ ,\ a_q)\ (\mathrm{mod}\ p)\ ,$$

and a *formal modular seminvariant in the Galois field* $GF[p^n]$ if, in this field, $P(A_0,\ A_1,\ \cdots,\ A_q)=P(a_0,\ a_1,\ \cdots,\ a_q)$, t being any mark of the $GF[p^n]$.

If $S_1,\ S_2,\ \cdots,\ S_r$ are a set of algebraic (formal modular) seminvariants of a binary form such that every algebraic (formal modular) seminvariant, which is by definition a polynomial, is equal (congruent, mod p, or equal in the field) to $S_1^rQ(S_1,\ S_2,\ \cdots,\ S_r)$, where Q is a polynomial in its argu-

* The investigation upon which this article is based was in part supported by a grant from the Heckscher Foundation for the Advancement of Research, established by August Heckscher at Cornell University. A preliminary report on this subject was presented to the Society, September 8, 1922, under the title *Fundamental systems of protomorphic formal modular seminvariants of binary forms*. This paper was received by the editors in October, 1925.

ments and r is an integer (positive, negative, or zero), $S_1, S_2, \cdots, S_r$ form a fundamental system of algebraic (formal modular) protomorphs of $f(x,y)$.

One set of algebraic protomorphs* of a form $f(x, y)$ of order q is $S_1, S_2, \cdots, S_q$, where

$$S_1 = a_0,$$

$$S_2 = a_0a_2 - a_1^2,$$

$$S_3 = a_0^2a_3 - 3a_0a_1a_2 + 2a_1^3,$$

$$S_4 = a_0a_4 - 4a_1a_3 + 3a_2^2,$$

$$(5) \qquad S_{2m} = a_0a_{2m} - \binom{2m}{1}a_1a_{2m-1} + \binom{2m}{2}a_2a_{2m-2} - \cdots$$

$$+ (-1)^{m-1}\binom{2m}{m-1}a_{m-1}a_{m+1} + \frac{1}{2}(-1)^m\binom{2m}{m}a_m^2,$$

$$S_{2m+1} = (a_0\theta - 2a_1)S_{2m}, \text{ where } \theta \text{ is the differential operator}$$

$$a_1\frac{\partial}{\partial a_0} + a_2\frac{\partial}{\partial a_1} + \cdots + a_q\frac{\partial}{\partial a_{q-1}}.$$

In §II of the present paper we find a fundamental system of formal modular protomorphs, mod p, of $f(x, y)$; in §III we find a fundamental system of formal modular protomorphs, mod p, of s binary forms.

These fundamental systems are remarkable in that they contain no member, not congruent, mod p, to an algebraic seminvariant except

$$\beta = \prod_{t=0}^{p-1}(a_0t + a_1) \equiv a_1^p - a_0^{p-1}a_1 \pmod{p}.$$

The problem of finding a finite set of formal seminvariants, mod p, of $f(x, y)$ such that every formal seminvariant is a polynomial in the members of this finite set is one of considerable difficulty especially as for a given form the minimum number of members of such a set or *fundamental system* may vary with p.† In the solution of this problem a fundamental rôle is played by three kinds of seminvariants; first, those which are congruent, mod p, to algebraic seminvariants; second, those which are expressible as sums of certain powers of linear expressions in the coefficients of $f(x, y)$, and third, those which are expressible as products of linear expressions in the coefficients, such as β.

For example, every seminvariant, mod p $(p>2)$ of the binary quadratic, $ax^2+2bxy+cy^2$, is congruent, mod p, to a polynomial in seminvariants

* Elliott, *Algebra of Quantics*, pp. 212-215.

† Williams, these Transactions, vol. 22 (1921), p. 56.

congruent, mod p, to the algebraic seminvariants a, $\Delta = ac - b^2$, and the formal modular seminvariants

$$\beta = \prod_{t=0}^{p-1} (at+b)$$

and

$$\gamma_0 = \prod_{t=0}^{p-1} (at^2+2bt+c)\,.^*$$

The case of the binary cubic is more complicated, but for $p=5$ the binary cubic $ax^3+3bx^2y+3cxy^2+dy^3$ has as a fundamental system† of formal modular seminvariants

(1) the four algebraic seminvariants

$$a\ ,\ \Delta\ ,\ S_3 = a^2d-3abc+2b^3\ ,\ \text{and}\ D = a^2d^2-6abcd+4b^3d+4ac^3-3b^2c^2,$$

(2) two seminvariants of the product type

$$\gamma_0 \text{ and } \delta_{00} = \prod_{t=0}^{4} (at^3+3bt^2+3ct+d),$$

(3) six seminvariants of the sum type

$$\sum_{t=0}^{4} R^r S^s, \quad \text{where } R = at^2+2bt+c, \quad S = at^3+3bt^2+3ct+d\ ,$$

$$r,s = 0,3;\quad 0,4;\quad 2,3;\quad 2,4;\quad 3,3;\quad 3,4.$$

Although no proof has been given that every formal modular seminvariant is a polynomial in seminvariants of these three types, no exception to such a theorem is known. In §IV we prove that certain seminvariants can be expressed as polynomials in seminvariants of these three types.

Section V is devoted to a brief discussion of the extension of the theorems of the previous articles to the Galois field $GF[p^n]$.

II

If $S_1, S_2, \cdots, S_q$ are the set of protomorphs in §I, and $a_0 \neq 0$, then

$$(1) \qquad a_2 = \frac{a_1^2+S_2}{a_0}\ ,$$

$$a_3 = \frac{a_1^3+3a_1S_2+S_3}{a_0^2}\ ,$$

$$a_i = \frac{P_i(a_1\ ,\ S_2\ ,\ S_3\ ,\ \cdots\ ,\ S_i)}{a_0^{i-1}} \qquad (i=4,\ 5,\ \cdots\ ,\ q),$$

* Dickson, *Madison Colloquium Lectures*, pp. 42 et seq.

† Williams, loc. cit., pp. 69-73.

where P is a polynomial with integral coefficients in its arguments and p is a prime such that the binomial coefficients $\binom{q}{j} \not\equiv 0$, mod p* ($j = 1, 2, \cdots, q-1$); then any formal modular seminvariant (which is by definition a polynomial in $a_0, a_1, a_2, \cdots, a_q$)

$$S(a_0, a_1, \cdots, a_q) = S\left[a_0, a_1, \frac{a_1^2 + S_2}{a_0}, \cdots, \frac{P(a_1, S_2, \cdots, S_q)}{a_0^{q-1}}\right]$$

$$= F\left(a_0, \frac{S_2}{a_0}, \frac{S_3}{a_0^2}, \cdots, \frac{S_q}{a_0^{q-1}}\right) + a_0^k G(a_0, a_1, S_2, \cdots, S_q),$$

where k is an integer, positive, negative, or zero, G is a polynomial in its arguments, and F includes all the terms of S not involving a_1 explicitly. If we write

$$F\left(a_0, \frac{S_2}{a_0}, \cdots, \frac{S_q}{a_0^{q-1}}\right) = \frac{1}{a_0^r}\Phi(a_0, S_2, \cdots, S_q)$$

where Φ is a polynomial in its arguments, we derive the equality $a_0^\lambda G = a_0^\mu S - a_0^\nu \Phi$, where λ, μ, ν are integers $\geqq 0$. Since S and Φ are seminvariants, G is also a seminvariant. Furthermore, G is from the manner of its formation divisible by a_1 and hence by

$$\beta = \prod_{i=0}^{p-1} (a_0 i + a_1) \equiv a_1^p - a_0^{p-1} a_1 \pmod{p}.$$

Treating the seminvariant G/β in like manner, we see that $S = a_0^q P$, where q is an integer, positive, negative, or zero, and P is a polynomial in $a_0, S_2, S_3, \cdots, S_q$. Hence we have proved the

THEOREM. *The seminvariants $S_i (i = 1, 2, \cdots, q)$ and β form a fundamental system of protomorphs of the binary q-ic form,* mod p, *p being a prime such that* $\binom{q}{j} \not\equiv 0$, mod p ($j = 1, 2, \cdots, q-1$).

If the coefficients of $f(x, y)$ are integers, reduced mod p (or elements of any Galois field) a polynomial in the coefficients, invariant under the group of transformations considered in this paper, is called a modular seminvariant. Since the seminvariant β vanishes for every set of integral values of a_0, a_1 we have the following corollary.

COROLLARY. *Corresponding to every modular seminvariant S of a binary form none of whose prefixed binomial coefficients is divisible by p there is an*

* $\binom{q}{j} = q(q-1)\cdots(q-j+1)/j!$; if any $\binom{q}{j} \equiv 0$, mod p, our form is a special form.

integer $\lambda \geqq 0$ (*and hence an infinite number*) *such that* $a_0^{\lambda} S$ *is congruent*, mod p, *to a polynomial in the algebraic protomorphs.*

This corollary is closely related to Miss Hazlett's Corollary I, p. 195, American Journal of Mathematics, vol. 43 (1921).

III

Definition of a fundamental system of protomorphs of s binary forms: *Given s binary forms of order $q_i (i = 1, 2, \cdots, s)$*

$$a_0^{(i)} x^{q_i} + \binom{q_i}{1} a_1^{(i)} x^{q_i - 1} y + \cdots + a_{q_i}^{(i)} y^{q_i}$$

and p a modulus not a factor of any $\binom{q_i}{j}$ $(j = 1, 2, \cdots, q_i - 1)$, we define $K_1, K_2, \cdots, K_\lambda$, a finite number of seminvariants of the s forms, to be a fundamental system of protomorphs of the s forms if they have the property that any seminvariant of the s forms is a rational function of $K_1, K_2, \cdots, K_\lambda$.

It is a known fact that the $\sum_{i=1}^{s} q_i$ seminvariants*

$$S_1^{(i)}, S_2^{(i)}, \cdots, S_{q_i}^{(i)} \qquad (i = 1, 2, \cdots, s)$$

and the $(s-1)$ joint seminvariants

$$J_i = a_0 a_1^{(i)} - a_1 a_0^{(i)} \qquad (i = 2, 3, \cdots, s)$$

have the property that any algebraic seminvariant of our s forms is a rational function of these. In the following theorem we prove the fact, which seems very remarkable, that these algebraic seminvariants together with the single formal modular seminvariant

$$\beta_1 = \prod_{t=0}^{p-1} (a_0^{(1)} t + a_1^{(1)}) \equiv [a_1^{(1)}]^p - [a_0^{(1)}]^{p-1} a_1^{(1)} \pmod{p},$$

form a fundamental system of formal modular seminvariants of the s forms, viz.

THEOREM. *The $\sum_{i=1}^{s} q_i$ seminvariants $S_1^{(i)}, S_2^{(i)}, \cdots, S_{q_i}^{(i)}$ $(i = 1, 2, \cdots, s)$ where $S_1^{(i)} = a_0^{(i)}$, $S_2^{(i)} = a_0^{(i)} a_2^{(i)} - a_1^{(i)2}$, etc. as in* §I, *the $(s-1)$ seminvariants $J_i = a_0^{(1)} a_1^{(i)} - a_1^{(1)} a_0^{(i)}$ $(i = 2, 3, \cdots, s)$, and the single seminvariant*

$$\beta_1 = \prod_{t=0}^{p-1} (a_0^{(1)} t + a_1^{(1)}) \equiv [a_1^{(1)}]^p - [a_0^{(1)}]^{p-1} a_1^{(1)} \pmod{p},$$

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* By $S_1^{(i)}, S_2^{(i)}, \cdots, S_{q_i}^{(i)}$ we mean the set of protomorphs in §I for the ith of our s forms.

are a fundamental system of formal modular protomorphs, mod p, *of the* s *binary forms*

$$a_0^{(i)}x^{q_i}+\binom{q}{1}a_1^{(i)}x^{q_i-1}y+\cdots+a_{q_i}^{(i)}y^{q_i} \quad (i=1,\ 2,\ \cdots,\ s)$$

p *being a prime such that no* $\binom{q_i}{j}\equiv 0$, mod p.

Proof. As in §II,

$$a_2^{(i)}=\frac{[a_1^{(i)}]^2+S_2^{(i)}}{a_0^{(i)}},$$

$$a_3^{(i)}=\frac{[a_1^{(i)}]^3+3a_1^{(i)}S_2^{(i)}+S_3^{(i)}}{a_0^{(i)2}},$$

$$a_j^{(i)}=P(a_1^{(i)},\ S_1^{(i)},\ S_2^{(i)},\ \cdots,\ S_j^{(i)}) \quad (i=1,\ 2,\ \cdots,\ s\ ;\ j=4,\ 5,\ 6,\ \cdots,q_i)$$

where P is a polynomial with integral coefficients in its arguments; furthermore,

$$a_1^{(i)}=J_i+\frac{a_1^{(1)}a_0^{(i)}}{a_0^{(1)}}.$$

Making the above substitutions in any seminvariant

$$S(a_0^{(1)},\ a_1^{(1)},\ \cdots,\ a_{q_1}^{(1)};\ \cdots;\ a_0^{(s)},\ a_1^{(s)},\ \cdots,\ a_{q_s}^{(s)}),$$

and separating into parts as in §II, we have

$$S=F(S_1^{(1)},\ \cdots,\ S_{q_s}^{(s)};\ J_1,\ J_2,\ \cdots,\ J_s)+a_0^{(1)\mu_1}a_0^{(2)\mu_2}\cdots a_0^{(s)\mu_s}$$

$$\times G(a_1^{(1)},\ S_1^{(1)},\ \cdots,\ J_1,\ \cdots),$$

$\mu_1,\ \mu_2,\ \cdots,\ \mu_s$ being integers and G a polynomial in $a_1^{(1)}$ and the seminvariants $S_j^{(i)}$, J_k. As in §II, G is a seminvariant of the system, divisible by $a_1^{(1)}$ and hence by β_1; the theorem follows by induction.

An example of the protomorphic representation of joint seminvariants is here given. In the seminvariant, mod 5, of

$$a_1x^2+2b_1xy+c_1y^2 \text{ and } a_2x^2+2b_2xy+c_2y^2,$$

$$S=-\sum_{t=0}^{4}(a_1t^2+2b_1t+c_1)^4(a_2t^2+2b_2t+c_2)^2$$

$$\equiv a_1^2a_2^2c_1^2+3a_1a_2^2b_1^3c_1+a_2^2b_1^4+a_2^2c_1^4+a_1^2a_2b_1b_2c_1+3a_1a_2b_1^3b_2$$

$$+2a_2b_1b_2c_1^3+3a_1^2b_1^2a_2c_2+3a_1^3a_2c_1c_2+a_1^3b_2^2c_1+a_1^2b_1^2b_2^2+3a_1a_2c_1^3c_2$$

$$+3a_2b_1^2c_1^2c_2+a_1b_1^2c_1^3+b_1^2b_2^2c_1^2+2a_1^2b_1b_2c_2+a_1b_1b_2c_1^2c_2+a_1^4c_2^2$$

$$+3b_1^3b_2c_1c_2+a_1^2c_1^2c_2^2+3a_1b_1^3c_1c_2^2+b_1^4c_2^2 \qquad \pmod 5,$$

substitute

$$b_2=\frac{J+a_2b_1}{a_1}, \qquad c_1=\frac{b_1^3-\Delta_1}{a_1},$$

$$c_2=\frac{J^2+2a_2b_1J+a_2^2b_1^2-a_1^2\Delta_2}{a_1^2a_2},$$

where $\Delta_1=b_1^2-a_1c_1$, $\Delta_2=b_2^2-a_2c_2$ and $J=a_1b_2-a_2b_1$. After making the above substitutions and collecting terms, we find that

$$S=\frac{1}{a_1^4a_2^2}(a_1^4a_2^4\Delta_1^2+a_1^4\Delta_1^4+a_1^4a_2^2\Delta_1J^2+3a_1^6a_2^2\Delta_1\Delta_2+a_2^2\Delta_1^3J^2$$

$$+3a_1^2a_2^2\Delta_1^3\Delta_2+a_1^4J^4+a_1^3\Delta_2^2+3a_1^6\Delta_2J^2+\Delta_1^2J^4$$

$$+a_1^4\Delta_1^3\Delta_2^2+3a_1^2\Delta_1^2\Delta_2J^2+4\beta_1a_2^3\Delta_1J+4\beta_1a_2J^3+\beta_1a_1^3a_2\Delta_2J),$$

where

$$\beta_1=b_1^5-a_1^4b_1 .$$

A much simpler case is that of

$$\beta_2=b_2^p-a_2^{p-1}b_2,$$

which is easily reduced to

$$\frac{J^p-a_1^{p-1}a_2^{p-1}J+a_2^p\beta_1}{a_1^p},$$

whence we derive the syzygy

$$a_1^p\beta_2-a_2^p\beta_1+a_1^{p-1}a_2^{p-1}J+J^p\equiv 0 \qquad \pmod p.$$

IV

A necessary and sufficient condition that a polynomial in the coefficients of a binary q-ic form be a formal seminvariant, mod p, of that form is,

obviously, that it be unaltered, mod p, by the substitution T on $a_0, a_1, \cdots, a_q$:

$$(1) \qquad \begin{pmatrix} a_0 & a_1 & \cdots & a_q \\ a_0 & a_0+a_1 & \cdots & a_0+\binom{q}{1}a_1+\binom{q}{2}a_2+\cdots+a_q \end{pmatrix},$$

p not being a factor of any $\binom{q}{k}$ in f.

Let

$$(2) \qquad R_j(t)=a_0t^j+\binom{j}{1}a_1t^{j-1}+\cdots+a_j \qquad (j=0, 1, 2, \cdots, q).$$

Then

$$R_j(0)=a_j, \quad \text{and} \quad TR_j(0)=R_j(1),$$

and in general

$$T^mR_j(0)=R_j(m) \qquad (m=1, 2, \cdots, p).$$

In particular

$$T^pR_j(0)=R_j(p)=R_j(0) .$$

Hence if for $R_i(t)$ we write R_i,

$$\sum_{i=0}^{p-1} R_0^{v_0}R_1^{v_1}\cdots R_q^{v_q},$$

where $v_0, v_1, \cdots, v_q$ are positive integers or zero, is a seminvariant.

Such a seminvariant we shall call a σ-seminvariant; a σ-seminvariant may, of course, be congruent to zero, mod p.

$$\Pi_1=\beta=\prod_{i=0}^{p-1} R_1(t)\equiv a_1^p-a_0^{p-1}a_1 \pmod{p},$$

the very important seminvariant which we have already met, is a particular case of

$$\Pi_j=\prod_{i=0}^{p-1} R_j(t)\equiv a_i^p+\cdots \pmod{p} \qquad (j=1, 2, \cdots, q).$$

These j seminvariants of a binary form we call π-seminvariants.

Now let

$$(3) \qquad S=\sum K_{k_0,k_1,\cdots,k_q}\, a_0^{k_0}a_1^{k_1}\cdots a_q^{k_q}$$

be any formal seminvariant, mod p, of a binary form f of order q with prefixed binomial coefficients no one of which is divisible by p, and let σ be a seminvariant of the kind which we call σ-seminvariants, viz.

$$(4) \qquad \sigma=\sum_{i=0}^{p-1} [R_0(t)]^{v_0}\cdots[R_q(t)]^{v_q},$$

where

$$R_0(t)=a_0, \; R_1(t)=a_0t+a_1, \; \cdots, \; R_q(t)=a_0t^q+\cdots+a_q.$$

Multiplying S by

$$a_0^{v_0} a_1^{v_1} \cdot \cdot \cdot a_q^{v_q}$$

and expanding, we have

$$a_0^{v_0} a_1^{v_1} \cdot \cdot \cdot a_q^{v_q} S = \sum L_{l_0, l_1, \cdots, l_q} a_0^{l_0} a_1^{l_1} \cdot \cdot \cdot a_q^{l_q}. \tag{5}$$

By use of the syzygies

$$a_1^p \equiv \beta + a_0^{p-1} a_1 \pmod{p},$$

$$a_j^p \equiv \Pi_j + \cdot \cdot \cdot \pmod{p} \ (j = 2, 3, \cdot \cdot \cdot, q),$$

we may express

$$\sum L_{l_0, l_1, \cdots, l_q} a_0^{l_0} a_1^{l_1} \cdot \cdot \cdot a_q^{l_q}$$

in the form

$$\sum M_{m_0, m_1, \cdots, m_q} a_0^{m_0} a_1^{m_1} \cdot \cdot \cdot a_q^{m_q} Q_{m_0, m_1, \cdots, m_q}(\beta_1, \Pi_1, \cdot \cdot \cdot, \Pi_q).$$

We shall then have the equality

$$a_0^{v_0} a_1^{v_1} \cdot \cdot \cdot a_q^{v_q} S \equiv \sum M_{m_0, m_1, \cdots, m_q} a_0^{m_0} a_1^{m_1} \cdot \cdot \cdot a_q^{m_q} Q_{m_0, m_1, \cdots, m_q} \pmod{p} \tag{6}$$

where each of $m_1, m_2, \cdots, m_q$ is less than p and the Q's are polynomials (with integral coefficients) in $\beta_1, \Pi_2, \cdots, \Pi_q$. The relation (6) and the $p-2$ congruences obtained by applying to (6) the substitution (1) $p-2$ times may be combined in the one relation

$$[R_0(t)]^{v_0} [R_1(t)]^{v_1} \cdot \cdot \cdot [R_q(t)]^{v_q} S \equiv \sum M_{m_0, m_1, \cdots, m_q} [R_0(t)]^{m_0} \cdot \cdot \cdot$$

$$\times [R_q(t)]^{m_q} Q_{m_0, \cdots, m_q} \pmod{p} \ (t = 0, 1, 2, \cdots, p-1). \tag{7}$$

Adding the p congruences represented by (7) we have, since S and the Q's are seminvariants, and since σ has the value given in (4),

$$\sigma S \equiv \sum M_{m_0, m_1, \cdots, m_q} P_{m_0, m_1, \cdots, m_q} Q_{m_0, m_1, \cdots, m_q} \pmod{p}, \tag{8}$$

where

$$P_{m_0, m_1, \cdots, m_q} \equiv a_0^{m_0} \sum_{t=0}^{p-1} [R_0(t)]^{m_1} \cdot \cdot \cdot [R_q(t)]^{m_q} \pmod{p}; \tag{9}$$

since each of $m_1, m_2, \cdots, m_q$ is less than p,

$$[R_0(t)]^{m_1} \cdot \cdot \cdot [R_q(t)]^{m_q}$$

represents a finite number μ of seminvariants, S_i $(i = 1, 2, \cdots, \mu)$. We have thus proved the

THEOREM. *The product of any formal seminvariant S of a binary form f of order q, with prefixed binomial coefficients with respect to any prime p which is not a factor of any of the binomial coefficients* $\binom{q}{1}, \binom{q}{2}, \cdots, \binom{q}{q-1}$,

by any σ-seminvariant is a polynomial in a_0 and a finite number of σ-seminvariants and π-seminvariants.

This theorem may also be stated thus:

THEOREM. *The seminvariants a_0, Π_i $(i=1, 2, \cdots, q)$ and S_j $(j=1, 2, \cdots, \mu)$ form a finite set of protomorphs of the binary form f.*

We have thus shown how to express any formal seminvariant S fractionally in terms of a finite number of seminvariants in a variety of ways. Some examples of the application of the above method follow.

Since

$$-\sum_{t=0}^{4}(b+at)^4 \equiv a^4 \pmod 5,$$

and

$$D=a^2d^2-6abcd+4b^3d+4ac^3-3b^2c^2$$

is an (algebraic) invariant of the binary cubic, let $\sigma=a^4$ and $S=D$. Applying the method which has just been described, we write down

$$-b^4D=-a^2b^4d^2+6ab^5cd-4b^7d-4ab^4c^3+3b^6c^2; \tag{10}$$

since $b^5\equiv\beta+a^4b$, mod 5,

$$-b^4D=-a^2b^4d^2-4acd\beta-4a^5bcd-4b^2d\beta-4a^4b^3d-4ab^4c^3 \pmod 5. \tag{11}$$

Making the four seminvariant substitutions and adding, we have

$$\begin{aligned}
-a^4D\equiv{}& a^2\sum_{t=0}^{4}(at+b)^4(at^3+3bt^2+3ct+d)\\
&+4a\beta\sum_{t=0}^{4}(at^2+2bt+c)\,(at^3+3bt^2+3ct+d)\\
&+4a^5\sum_{t=0}^{4}(at+b)\,(at^2+2bt+c)\,(at^3+3bt^2+3ct+d)\\
&+4\beta\sum_{t=0}^{4}(at+b)^2(at^3+3bt^2+3ct+d)\\
&+4a^4\sum_{t=0}^{4}(at+b)^3(at^3+3bt^2+3ct+d)\\
&+4a\sum_{t=0}^{4}(at+b)^4(at^2+2bt+c)^3\\
&+2\beta\sum_{t=0}^{4}(at+b)\,(at^2+2bt+c)^2\\
&+2a^4\sum_{t=0}^{4}(at+b)^2(at^2+2bt+c)^2 \pmod p.
\end{aligned} \tag{12}$$

Thus not only are all seminvariants, whether congruent to algebraic seminvariants or not, expressible fractionally in terms of algebraic seminvariants and β, but in turn all seminvariants, whether congruent to algebraic seminvariants or not, are expressible in terms of a and a finite number of seminvariants of the σ and π forms. By choosing $\sigma = a^{p-1} \equiv -\sum_{t=0}^{p-1} (at+b)^{p-1}$ we may express $a^{p-1}S$, where S is any seminvariant, in like manner. Since

$$-\sum_{t=0}^{p-1} (at^2+2bt+c)^{p-1} \equiv a^{p-1}+\Delta^{(p-1)/2} \pmod{p}, \tag{13}$$

where $\Delta = b^2-ac (p>2)$, multiplication of any seminvariant S by c^{p-1} and application of this method gives an expression of $(a^{p-1}+\Delta^{(p-1)/2})S$ in terms of a and σ- and π-seminvariants; subtracting the expression for $a^{p-1}S$ we have an expression for $\Delta^{(p-1)/2}S$ in terms of a and σ- and π-seminvariants. Setting $S = \Delta^{\gamma}$ ($\gamma = 1, 2, \cdots$) and applying the present method we have the

THEOREM. *Δ^{δ}, where $\delta \geqq (p-1)/2$, is congruent to a polynomial in a the seminvariants β and Π_2 and the seminvariants*

$$\sum_{t=0}^{p-1} [R_1(t)]^{v_1} [R_2(t)]^{v_2} \text{ where } v_1, v_2 < p.$$

By using the congruence proved in the following lemma we may readily prove the

THEOREM. *If D is the discriminant of the form $ax^3+3bx^2y+3cxy^2+dy^3$ and p is a prime of the form $3m+1$, D^{δ}, where $\delta \geqq (p-1)/6$, is congruent,* mod p, *to a polynomial in a, the seminvariants Π_1, Π_2, Π_3 and $\sum_{t=0}^{p-1}[R_1(t)]^{v_1} \times [R_2(t)]^{v_2} [R_3(t)]^{v_3}$ (v_1, v_2, v_3 ranging over values $<p$).*

LEMMA. *If p is a prime of the form $3k+1$,*

$$a^{2(p-1)/3} - \sum_{t=0}^{p-1} (at^3+bt^2+3ct+d)^{2(p-1)/3} \equiv \binom{2(p-1)/3}{(p-1)/3} D^{(p-1)/6} \pmod{p},$$

where $D = a^2d^2-6abcd+4b^3d+4ac^3-3b^2c^2$, the discriminant of $ax^3+3bx^2y+3cxy^2+dy^3$.

Proof.

$$at^3+3bt^2+3ct+d = \frac{1}{a^2}[(at+b)^3+3\Delta(at+b)+S_3],$$

where $\Delta = ac - b^2$, $S_3 = a^2d - 3abc + 2b^3$ and $a \neq 0$. Since $\sum_{t=0}^{p-1} t^k \equiv 0$, mod p, when k is 0 and when k is not a multiple of $p-1$ and $\sum_{t=0}^{p-1} t^k \equiv -1$, when k is a non-zero multiple of $p-1$,

$$a^{2(p-1)/3} - \sum_{t=0}^{p-1} (at^3+3bt^2+3ct+d)^{2(p-1)/3} \equiv + [\text{coefficient of } t^{p-1} \text{ in the expansion}$$

$$\text{of } (at^3+3bt^2+3ct+d)^{2(p-1)/3}] \equiv +[\text{coefficient of } t^{p-1} \text{ in the expansion of} \frac{1}{a^{4(p-1)/3}}$$

$$\times \{(at+b)^3+3\Delta(at+b)+S_3\}^{2(p-1)/3}] \equiv +\{\text{coefficient of } t^{p-1} \text{ in the expansion of}$$

$$\frac{1}{a^{4(p-1)/3}} (S_3+3\Delta at+a^3t^3)^{2(p-1)/3} + \frac{b}{a^{4(p-1)/3}} P(a,b,c,d)\}, \text{ mod } p, \text{ where } P(a,b,c,d)$$

is a polynomial.

Now the coefficient of t^{p-1} in the expansion of $(S_3+3\Delta at+a^3t^3)^{2(p-1)/3}$

$$\equiv a^{p-1} \sum_{m=0}^{(p-1)/6} \frac{\left[\frac{2}{3}(p-1)\right]!}{\left[\frac{p-1}{3}-m\right]!\left[\frac{p-1}{3}-2m\right]![3m]!} S_3^{(p-1)/3-2m} \Delta^{3m} 3^{3m} \pmod{p},$$

where $0! = 1$.

For any m from 0 to $(p-1)/6$*

$$\frac{\left[\frac{2}{3}(p-1)\right]! 3^{3m}}{\left[\frac{p-1}{3}-m\right]!\left[\frac{p-1}{3}-2m\right]![3m]!}$$

$$\equiv \frac{\left[\frac{2}{3}(p-1)\right]!\left[\frac{p-1}{6}\right]! 2^{2m}}{\left[\frac{1}{3}(p-1)\right]!\left[\frac{1}{3}(p-1)\right]!\left[\frac{p-1}{6}-m\right]![m]!} \pmod{p},$$

a relation which may be proved by noticing that it is equivalent to the relation

$$\frac{3^{3m}\left[\frac{p-1}{6}-m\right]! m!\left[\frac{p-1}{3}\right]!\left[\frac{p-1}{3}\right]!}{\left[\frac{p-1}{3}-m\right]!\left[\frac{p-1}{3}-2m\right]![3m]! 2^{2m}} \equiv 1 \pmod{p}.$$

* For the following elegant proof of this relation I am indebted to Professor W. A. Hurwitz

This last relation is evidently true when $m=0$; and for $0 \leqq m \leqq (p-1)/6$ when m is replaced by $m+1$, its left hand side is multiplied by

$$\frac{3^3(m+1)((p-1)/3-m)((p-1)/3-2m)((p-1)/3-2m-1)}{2^2((p-1)/6-m)(3m+1)(3m+2)(3m+3)},$$

which is readily proved congruent to unity, mod p. Hence

$$\begin{aligned} a^{p-1}\sum_{m=0}^{(p-1)/6} & \frac{[\tfrac{2}{3}(p-1)]!}{[(p-1)/3-m]![(p-1)/3-2m]![3m]!} S_3^{(p-1)/3-m}\Delta^{3m}3^{3m} \\ & \equiv \binom{2(p-1)/3}{(p-1)/3} a^{p-1}\sum_{m=0}^{(p-1)/6}\binom{(p-1)/6}{m}(S_3^2)^{(p-1)/6-m}(4\Delta^3)^m \\ & \equiv \binom{2(p-1)/3}{(p-1)/3} a^{p-1}(S_3^2+4\Delta^3)^{(p-1)/6} \\ & \equiv a^{4(p-1)/3}\binom{2(p-1)/3}{(p-1)/3} D^{(p-1)/6} \pmod{p}, \end{aligned}$$

the last congruence following from the algebraic identity

$$a^2D=S_3^2+4\Delta^3.$$

We have then proved that

$$a^{2(p-1)/3}-\sum_{t=0}^{p-1}(at^3+3bt^2+3ct+d)^{2(p-1)/3} \equiv \binom{2(p-1)/3}{(p-1)/3} D^{(p-1)/6}+bP(a, b, c, d) \pmod{p}.$$

It is well known that the left hand side of this last congruence is a formal modular invariant; we now proceed to prove that it is absolutely isobaric. Obviously

$$(at^3+3bt^2+3ct+d)^{2(p-1)/3}=\sum_{i=0}^{2(p-1)} P_i(a, b, c, d)t^i.$$

If we assign to a, b, c, d the weights 0, 1, 2, 3 respectively and to t the weight 1 and expand the left hand side, each term of this expansion will have the weight $2(p-1)$ and hence each $P_i(a, b, c, d)\,t^i$ is absolutely isobaric and of weight $2(p-1)$, i. e. each $P_i(a, b, c, d)$ is absolutely isobaric and has the weight $2(p-1)-i$; this last is, of course, quite independent of t and from this point on we do not use the fact that we have assigned a weight to t. Now consider the following relation:

$$\begin{aligned} a^{2(p-1)/3}-\sum_{t=0}^{p-1}(at^3+3bt^2+3ct+d)^{2(p-1)/3} & = a^{2(p-1)/3}+a^{2(p-1)/3}\sum_{t=0}^{p-1} t^{2(p-1)} \\ +P_{2p-3}(a, b, c, d)\sum_{t=0}^{p-1} t^{2p-3} & +\cdots+P_{p-1}(a, b, c, d)\sum_{t=0}^{p-1} t^{p-1} \\ +\cdots+pP_0(a, b, c, d) & \equiv -P_{p-1}(a, b, c, d) \pmod{p}, \end{aligned}$$

since

$$\sum_{t=0}^{p-1} t^k \equiv 0, \text{ mod } p, \quad k=0 \text{ and } \not\equiv 0, \text{ mod } p-1,$$

$$\equiv -1, \text{ mod } p, \ k \text{ a non-zero multiple of } p-1.$$

We see therefore that each member of our congruence except $bP(a, b, c, d)$ is an absolutely isobaric formal modular invariant of weight $p-1$; accordingly, the same must be true of this remaining term unless it is congruent to zero, mod p. But if $bP(a, b, c, d)$ is an invariant divisible by b, it is also divisible, mod p, by $a+b$, whence

$$bP(a, b, c, d) \equiv (a+b)Q(a, b, c, d) \pmod{p},$$

where Q is a polynomial. But $bP(a, b, c, d)$, if it is $\not\equiv 0$, mod p, is absolutely isobaric and of weight $p-1$. Hence $(a+b)Q(a, b, c, d)$ is absolutely isobaric, and hence $Q(a, b, c, d)$ is not absolutely isobaric. Then let

$$Q(a, b, c, d) = Q_0(a, b, c, d) + Q_1(a, b, c, d) + \cdots + Q_k(a, b, c, d),$$

where each $Q_i(a, b, c, d) \not\equiv 0$, mod p, and is absolutely isobaric, Q_0 containing the terms of least weight, Q_1 those of next highest weight, etc. Then

$$bP(a, b, c, d) \equiv (a+b)Q(a, b, c, d) \equiv aQ_0 + \cdots + bQ_k \pmod{p}.$$

Since the weight of Q_0 is $<$ the weight of Q_k, the weight of aQ_0 is $<$ the weight of bQ_k which contradicts the hypothesis that $bP(a, b, c, d)$ is absolutely isobaric of weight $p-1$. Hence $P \equiv 0$, mod p.

Thus the lemma has been proved for all a, b, c, d for which $a \not\equiv 0$. For $a=0$, its truth may be verified directly.

Not every algebraic seminvariant of a binary form is congruent, with respect to some prime, to a polynomial in a and the σ- and π-seminvariants of that form. For example, if $p \geqq 7$, Π_1 and Π_2 are each of degree 7 and any σ-seminvariant

$$\sigma = \sum_{t=0}^{p-1} a^{v_0}(at+b)^{v_1}(at^2+2bt+c)^{v_2}$$

is of degree 3, since

$$\sum_{t=0}^{p-1} t^\lambda \equiv 0 \pmod{p},$$

for all $\lambda < p-1$. Hence any seminvariant of degree 2 with respect to any prime $\geqq 7$, which is a polynomial in a and the σ- and π-seminvariants of the binary quadratic is congruent to a numerical multiple of a^2; the algebraic seminvariant $ac-b^2$ is therefore not congruent, mod p ($p \geqq 7$) to a polynomial in a, and the σ- and the π-seminvariants of the binary quadratic. However, the interesting and difficult problem presents itself, as remarked

in I: Is every formal modular seminvariant, mod p, of a binary form congruent, mod p, to a polynomial in the algebraic seminvariants and the σ- and π-seminvariants of that form?

For the sake of simplicity of statement, attention has in this article been directed to seminvariants of one form only; the method of the present article is, however, applicable to the seminvariants of any number of forms, though we shall, of course, be confronted with what may be termed "mixed seminvariants" such as

$$\sum_{t=0}^{p-1} (a_0t^2+2a_1t+a_2)^{p-1}(b_0t^2+2b_1t+b_2)^{p-1},$$

where a_0, a_1, a_2 are coefficients of the form $(a_0, a_1, \cdots, a_{q_1})(x,y)^{q_1}$ and b_0, b_1, b_2 of the form $(b_0, b_1, \cdots, b_{q_2})\ (x, y)^{q_2}$.

The theorem for s forms is obviously as follows:

THEOREM. *The seminvariants of s binary forms $(a_0, a_1, \cdots, a_q)(x,y)^{q_i}$ $(i=1, 2, \cdots, s)$(every binomial coefficient $\not\equiv 0$, mod p) can be expressed fractionally in terms of $a_0^{(i)}$ $(i=1, 2, \cdots, s)$, σ-seminvariants, pure and mixed, and π-seminvariants of the s forms, the exponents of the σ-seminvariants all being $< p$ and the set consequently being finite.*

In the preceding sections we have, for simplicity, considered only formal invariants under linear non-singular transformations whose coefficients are integers reduced mod p. Our results are easily generalized; to obtain a generalization we consider as before a binary form of order q with prefixed binomial coefficients, but take the coefficients of our non-singular linear transformations to be elements of a $GF[p^n]$, where p is any prime not a factor of any of the prefixed binomial coefficients of the form.

Following the same method of proof as in §II we prove the

THEOREM. *The seminvariants S_i $(i=1, 2, \cdots, q)$ and $\beta=b^{p^n}-a^{p^n-1}b$ form a set of formal protomorphs of the binary q-ic form over the $GF[p^n]$.*

As in §III, we prove the theorem of that article; the fundamental system is changed in only one particular; we now have

$$\beta_1=\prod_t [a_0^{(1)}t+a_1^{(1)}]=[a_1^{(1)}]^{p^n}-[a_0^{(1)}]^{p^n-1}a_1,$$

the product being taken for all values of t in the $GF[p^n]$, and the equality being true in that field. As an example, the syzygy given in the last line of §III holds in the $GF[p^n]$, if

$$\beta_i=\prod_t [a_0^{(i)}t+a_1^{(i)}] \qquad (i=1, 2),$$

t ranging over all the elements of the $GF[p^n]$.

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NON-SYNCHRONIZED RELATIVE INVARIANT INTEGRALS*

BY

K. P. WILLIAMS

1. **Introduction.** Poincaré has devoted a large part of the third volume of his *Méthodes Nouvelles de Mécanique Céleste* to invariant integrals. This is sufficient to show their importance and the diversity of their application. One of the most interesting questions connected with the theory lies in searching for relationships between the different types into which one naturally divides invariant integrals. This paper is devoted to such a consideration.

We begin by recalling some of the fundamental ideas of the theory in the form in which they will be used here.

Suppose we have the system of differential equations

$$\frac{dz_i}{dt} = Z_i(z_1, \cdots, z_n) \qquad (i=1, 2, \cdots, n)$$

with solutions

$$z_i = \bar{\varphi}_i(t, a_1 \cdots, a_n) .$$

When $t=0$, we can in general choose the a_i so that z_i takes an arbitrary value. Now let C_0 be some curve in the n-dimensional continuum $z_1, \cdots, z_n$. We can determine the constants a_i as functions of a parameter α such that as α varies from α_1 to α_2 the point $z_1, \cdots, z_n$ will describe the curve C_0. We are thus led to consider the one-parameter family of trajectories

$$z_i = \varphi_i(t, \alpha), \qquad \alpha_1 \leqq \alpha \leqq \alpha_2 .$$

For every value of α we shall have a trajectory T_α. As t varies from the initial value $t=0$, we have a point on each trajectory, and we describe as *contemporaneous* the points on the various T_α that correspond to the same value of t. In this way we regard the differential equations as giving a certain continuous displacement to the arbitrarily chosen curve C_0, producing the family of curves C_t.

Suppose now that we consider the line integral

$$J = \int_{\alpha_1}^{\alpha_2} \sum_{i=1}^{n} L_i \delta z_i, \qquad L_i = L_i(z_1, \cdots, z_n),$$

* Presented to the Society, April 11, 1925; received by the editors in June, 1925.

along any curve C_t. The equation of the curve being $z_i=\varphi_i(t, a)$ where a is the variable parameter and t is constant for the curve, we have for the integral

$$J=\int_{a_1}^{a_2}\sum L_i\frac{\partial\varphi_i}{\partial a}da.$$

The integrand being a function of t and a, it is seen that J is a function of t in general. If it so happens that J is a constant for all the curves of the family C_t, and that irrespective of how C_0 was chosen, we say that J is an *absolute invariant integral.*

Let us consider again the solutions of the differential equations in the form

$$z_i=\bar{\varphi}_i(t, a_1, \cdots, a_n).$$

Along any curve C_t we have

$$\delta z_i=\frac{\partial\bar{\varphi}_i}{\partial a_1}\delta a_1+\cdots+\frac{\partial\bar{\varphi}_i}{\partial a_n}\delta a_n.$$

It is obvious that if J is an absolute invariant, and we substitute the values of z_i and δz_i in the form $\sum L_i\delta z_i$ the variable t must disappear, leaving a function of a_i, and δa_i. This observation enables us to pass to a more general invariant.

We have so far considered a path joining contemporaneous positions on the various trajectories. But we can consider a curve cutting across the trajectories in *non-contemporaneous points.* We are thus led to consider an invariant of the form

$$I=\int\sum L_i\delta z_i+K\delta t,$$

and are to determine K in terms of L_i and Z_i. Its value is seen to be

$$K=-\sum L_iZ_i$$

from the following considerations. Along a curve through non-contemporaneous positions we have

$$\delta z_i=\frac{\partial\bar{\varphi}_i}{\partial t}\delta t+\frac{\partial\bar{\varphi}_i}{\partial a_1}\delta a_1+\cdots+\frac{\partial\bar{\varphi}_i}{\partial a_n}\delta a_n,$$

whence,

$$\delta z_i-Z_i\delta t=\frac{\partial\bar{\varphi}_i}{\partial a_1}\delta a_1+\cdots+\frac{\partial\bar{\varphi}_i}{\partial a_n}\delta a_n.$$

Recalling what was said above concerning the result of substituting z_i and δz_i in $\sum L_i \delta z_i$ on the supposition that J was invariant along the curves C_t, we see that

$$\sum L_i(\delta z_i - Z_i \delta t)$$

will be turned into the same function when we are considering variations along any path cutting the trajectories. It is therefore a constant. We thus have two important types of invariant integrals, a restricted type and a more general one. The notable thing is that from the restricted type we have constructed the more general one. An invariant of the form

$$J = \int \sum L_i \delta z_i,$$

taken along a curve through contemporaneous positions on the trajectories we shall call a *synchronized* invariant integral, while one of the form

$$I = \int \sum L_i \delta z - L_i Z_i \delta t,$$

that is, taken along a curve that does not pass through contemporaneous positions, we shall call a *non-synchronized* invariant integral.

In addition to the absolute invariants so far considered we have the *relative* invariants. In this case J is invariant provided C_0 is any closed curve. We see at once a vast difference between this case and the former, for it is no longer necessary that the solutions of the differential equations shall render the integrand independent of t. For instance, if the integrand had a term of the form $\varphi(t)\delta\theta(a_1, \cdots, a_n)$ this term would be zero when taken around a closed curve, for $a_1, \cdots, a_n$ would have final values equal to their initial ones.

It is evident that we have to consider both the *synchronized* and the *non-synchronized relative invariant*. Using geometrical language we speak of a tube of trajectories and closed curves on it, which in the first case pass through contemporaneous positions, and in the second case, through non-contemporaneous positions, on the trajectories. A well known non-synchronized invariant is that of Cartan, which he derives in an indirect manner.*

It is our purpose to make a direct approach to the question of the relative invariant integrals of the first order, with a view of obtaining non-synchronized invariants from synchronized ones.†

* Cartan, *Leçons sur les Invariants Intégraux*, p. 4.

† The relative invariants of the first order are usually reduced to absolute invariants of the second order. See Whittaker, *Analytical Dynamics*, 1st edition, p. 265. Such a reduction requires

2. **The synchronized relative invariant.** We take the system of equations

$$\frac{dz_i}{dt} = Z_i ,$$

and assume that we have a tube of trajectories given by

$$z_i = \varphi_i (t, a), \qquad a_1 \leqq a \leqq a_2 ,$$

where

$$\varphi_i (t, a_1) = \varphi_i (t, a_2) .$$

Under what conditions will

$$J = \int_{a_1}^{a_2} \sum_{i=1}^{n} L_i \delta z_i = \int_{a_1}^{a_2} \sum L_i \frac{\partial z_i}{\partial a} da$$

be a relative invariant integral?

We have

$$\frac{dJ}{dt} = \int_{a_1}^{a_2} \sum \left(\frac{dL_i}{dt} \frac{\partial z_i}{\partial a} + L_i \frac{\partial^2 z_i}{\partial t \partial a} \right) da .$$

But

$$\int_{a_1}^{a_2} \sum L_i \frac{\partial^2 z_i}{\partial t \partial a} = \sum L_i \frac{\partial z_i}{\partial t} \Big|_{a_1}^{a_2} - \int_{a_1}^{a_2} \sum \frac{\partial z_i}{\partial t} \frac{\partial L_i}{\partial a} da$$

$$= \sum L_i Z_i \Big|_{a_1}^{a_2} - \int_{a_1}^{a_2} \sum_{k=1}^{n} Z_k \frac{\partial L_k}{\partial a} da$$

$$= - \int_{a_1}^{a_2} \sum_{i=1}^{n} \sum_{k=1}^{n} Z_k \frac{\partial L_k}{\partial z_i} \frac{\partial z_i}{\partial a} da,$$

the integrated term having the same value at the two limits. Also

$$\frac{dL_i}{dt} = \sum_{k=1}^{n} Z_k \frac{\partial L_i}{\partial z_k} .$$

Therefore

$$\frac{dJ}{dt} = \int_{a_1}^{a_2} \sum_{i=1}^{n} \frac{\partial z_i}{\partial a} \sum_{k=1}^{n} Z_k \left(\frac{\partial L_i}{\partial z_k} - \frac{\partial L_k}{\partial z_i} \right) da .$$

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the use of Stokes' Generalized Theorem, that is, generalized to n dimensions. The next paragraph shows that the relative invariant of first order can be easily treated directly. Since presenting this paper, my attention has been called to a note by Goursat in the Comptes Rendus, vol. 174 (1922), p. 1090. In this note Goursat calls attention to the fact that the *complete* invariants of Cartan, which I call synchronized, can be obtained from the less general ones. His treatment requires for its basis a complete theory for invariants of different order. The purpose of the present paper is to treat directly an interesting case without any elaborate machinery, and obtain in that way the important mechanical invariant of Cartan.

This will obviously be zero if

$$\sum_{k=1}^{n} Z_k\left(\frac{\partial L_i}{\partial z_k}-\frac{\partial L_k}{\partial z_i}\right)=\frac{\partial M}{\partial z_i},$$

where M is some function of $z_1, \cdots, z_n$. For, under this condition,

$$\frac{dJ}{dt}=M_{a_2}-M_{a_1},$$

and since the variables z_i resume for $a=a_2$ the values they possessed for $a=a_1$, we see that $dJ/dt=0$. We thus have

THEOREM 1. *The integral*

$$\int \sum L_i \delta z_i$$

will be a synchronized relative invariant of the system $dz_i/dt=Z_i$ *if there is a function* $M(z_1, \cdots, z_n)$ *such that*

$$\sum_{k=1}^{n} Z_k\left(\frac{\partial L_i}{\partial z_k}-\frac{\partial L_k}{\partial z_i}\right)=\frac{\partial M}{\partial z_i}.$$

3. **The non-synchronized relative invariant.** We can obtain a curve that passes through non-contemporaneous positions on the different trajectories by writing

$$t=\theta(a, \tau)$$

where τ is a new parameter. In particular we choose θ so that

$$\theta(a, 0)=0,$$
$$\theta(a_1, \tau)=\tau=\theta(a_2, \tau).$$

Such a choice will be effected by putting

$$t=\tau\cdot\theta_1(a)\sin\frac{a-a_1}{a_2-a_1}\pi+\tau.$$

We note that

$$\left.\frac{\partial t}{\partial \tau}\right|_{a=a_1}=\left.\frac{\partial t}{\partial \tau}\right|_{a=a_2}.$$

The curves on the tube of trajectories that we are now considering are given by

$$z_i=\bar{\varphi}_i(\tau, a)=\varphi_i(\theta(a, \tau), a).$$

We shall denote these curves by C_τ'. We see that $C_0'=C_0$. Also the trajectory T_{a_1} cuts C_τ' in the same point in which it cuts C_τ.

We shall now determine K so that

$$I=\int_{a_1}^{a_2}\sum_{i=1}^{n}\left[\bar{L}_i\frac{\partial\bar{z}_i}{\partial a}+\bar{K}\frac{\partial t}{\partial a}\right]da$$

is invariant. The bar indicates that all quantities are expressed in terms of τ and a.

We have

$$\frac{dI}{d\tau}=\int_{a_1}^{a_2}\sum\left[\frac{\partial\bar{L}_i}{\partial\tau}\left(\frac{\partial\bar{z}_i}{\partial a}\right)+\bar{L}_i\frac{\partial^2\bar{z}_i}{\partial a\partial\tau}+\frac{\partial\bar{K}}{\partial\tau}\frac{\partial t}{\partial a}+\bar{K}\frac{\partial^2 t}{\partial\tau\partial a}\right]da.$$

Evidently

$$\frac{\partial\bar{L}_i}{\partial\tau}=\sum_{k=1}^{n}\frac{\partial\overline{L}_i}{\partial z_k}\frac{\overline{\partial z_k}}{\partial t}\frac{\partial t}{\partial\tau}=\sum_{k=1}^{n}\overline{Z}_k\frac{\partial\overline{L}_i}{\partial z_k}\frac{\partial t}{\partial\tau},$$

$$\int_{a_1}^{a_2}\sum_{i=1}^{n}\bar{L}_i\frac{\partial^2\bar{z}_i}{\partial a\partial\tau}da=\sum_{i=1}^{n}\bar{L}_i\frac{\partial\bar{z}_i}{\partial\tau}\Big|_{a_1}^{a_2}-\int_{a_1}^{a_2}\sum_{i=1}^{n}\frac{\partial\bar{z}_i}{\partial\tau}\frac{\partial\bar{L}_i}{\partial a}da$$

$$=\sum_{i=1}^{n}\bar{L}_i\bar{Z}_i\frac{\partial t}{\partial\tau}\Big|_{a_1}^{a_2}-\int_{a_1}^{a_2}\sum_{k=1}^{n}\overline{Z}_k\frac{\partial\overline{L}_k}{\partial a}\frac{\partial t}{\partial\tau}da$$

$$=-\int_{a_1}^{a_2}\sum_{k=1}^{n}\sum_{i=1}^{n}\overline{Z}_k\frac{\partial\overline{L}_k}{\partial z_i}\frac{\partial\bar{z}_i}{\partial a}\frac{\partial t}{\partial\tau}da.$$

The integrated part has vanished since it has the same value at both limits. Further

$$\frac{\partial\bar{K}}{\partial\tau}=\sum_{i=1}^{n}\frac{\partial\overline{K}}{\partial z_i}\overline{Z}_i\frac{\partial t}{\partial\tau},$$

$$\int_{a_1}^{a_2}\bar{K}\frac{\partial^2 t}{\partial\tau\partial a}da=-\int_{a_1}^{a_2}\frac{\partial\bar{K}}{\partial a}\frac{\partial t}{\partial\tau}da.$$

Hence

$$\frac{dI}{d\tau}=\int_{a_1}^{a_2}\left[\sum_{i=1}^{n}\frac{\partial\bar{z}_i}{\partial a}\sum_{k=1}^{n}\overline{Z}_k\left(\frac{\partial\overline{L}_i}{\partial z_k}-\frac{\partial\overline{L}_k}{\partial z_i}\right)+\sum_{k=1}^{n}\frac{\partial\overline{K}}{\partial z_k}\overline{Z}_k\frac{\partial t}{\partial a}-\frac{\partial\bar{K}}{\partial a}\right]\frac{\partial t}{\partial\tau}da.$$

We shall assume that J is a synchronized invariant satisfying the relation

$$\sum_{k=1}^{n}Z_k\left(\frac{\partial L_i}{\partial z_k}-\frac{\partial L_k}{\partial z_i}\right)=\frac{\partial M}{\partial z_i}.$$

Therefore

$$\frac{dI}{d\tau}=\int_{a_1}^{a_2}\left[\sum_{i=1}^{n}\left(\frac{\overline{\partial M}}{\partial z_i}\frac{\partial \bar{z}_i}{\partial a}+\frac{\overline{\partial K}}{\partial z_i}Z_i\frac{\partial t}{\partial a}\right)-\frac{\partial \bar{K}}{\partial a}\right]\frac{\partial t}{\partial \tau}da$$

$$=\int_{a_1}^{a_2}\left[\frac{\partial}{\partial a}\left(\bar{M}-\bar{K}\right)+\frac{\partial t}{\partial a}\sum_{i=1}^{n}\frac{\overline{\partial K}}{\partial z_i}\bar{Z}_i\right]\frac{\partial t}{\partial \tau}da\ .$$

This will be zero if $K=M$, and if

$$\sum_{i=1}^{n}\frac{\overline{\partial K}}{\partial z_i}\bar{Z}_i=\sum_{i=1}^{n}\frac{\overline{\partial M}}{\partial z_i}\frac{\partial \bar{z}_i}{\partial t}=0\ ,$$

that is, if M is an "integral" of the system of differential equations. The necessity of the condition follows from the arbitrariness of $\partial t/\partial\tau$ and $\partial t/\partial a$.

The family of curves C' depends upon the choice of the function $\theta_1(a)$, but always includes the curve C_0. We can choose $\theta_1(a)$ so that C_τ' will include any designated curve that is drawn about the tube, as shown by the following considerations. Let $\bar{C}$ be such a curve. Let it be cut by the trajectory T_{a_1} at the point for which the value of t is t_{a_1}. Any other trajectory T_a will cut $\bar{C}$ at a definite point, for which the value of t is t_a. We have merely to determine $\theta_1(a)$ by the relation

$$t_a=t_{a_1}\cdot\theta_1(a)\sin\frac{a-a_1}{a-a_2}\pi+t_{a_1}.$$

This will determine θ_1 for every value of a except $a=a_1$ and $a=a_2$. For the latter values θ_1 is indeterminate, but can be chosen from continuity considerations.

Since the set of curves C_τ can be made to include the arbitrary curves $\bar{C}$ and always includes the curves C_0 we can state

THEOREM 2. *If*

$$\int\sum L_i\partial z_i$$

is a synchronized relative invariant around a tube of trajectories, then

$$\int\sum L_i\delta z_i+M\delta t\ ,$$

where M has the significance given in Theorem 1, *is a non-synchronized invariant, provided M is an "integral" of the original system of equations.*

4. **An example.** One naturally wishes a simple illustration of such a theory as has been sketched. We turn to the expression

$$\sum_{k=1}^{n} Z_k\left(\frac{\partial L_i}{\partial z_i}-\frac{\partial L_k}{\partial z_i}\right),$$

and seek to make it as simple as possible. Let n be even. Consider

$$\frac{\partial L_i}{\partial z_k}-\frac{\partial L_k}{\partial z_i}$$

for a fixed value of i. If

$$L_i=z_{n/2+i}, \qquad \text{when } 1\leqq i\leqq n/2,$$

$$L_i=0, \qquad \text{when } n/2<i\leqq n,$$

we see the expression above $=1$, when $1\leqq i\leqq n/2$ for the value $k=n/2+i$, and $=0$ for all other values of k. If, however, $n/2<i\leqq n$ the expression $=-1$, for $k=i-n/2$ and $=0$ for other values. Thus

$$\sum_{k=1}^{n} Z_k\left(\frac{\partial L_i}{\partial z_k}-\frac{\partial L_k}{\partial z_i}\right)=Z_{n/2+i}, \qquad \text{for } 1\leqq i\leqq n/2,$$

$$=-Z_{i-n/2}, \qquad \text{for } n/2<i\leqq n.$$

Suppose now that

$$Z_i=\frac{\partial}{\partial z_{n/2+i}}H(z_1, \cdots, z_n), \qquad \text{for } 1\leqq i\leqq n/2,$$

$$Z_i=-\frac{\partial H}{\partial z_{i-n/2}}, \qquad \text{for } n/2<i\leqq n.$$

We have then

$$\sum_{k=1}^{n} Z_k\left(\frac{\partial L_i}{\partial z_k}-\frac{\partial L_k}{\partial z_i}\right)=-\frac{\partial H}{\partial z_i}, \qquad 1\leqq i\leqq n,$$

and therefore have the synchronized relative invariant

$$\int \sum_{i=1}^{n/2} z_{n/2+i}\,\delta z_i,$$

for the Hamiltonian system

$$\frac{dz_i}{dt}=\frac{\partial H}{\partial z_{n/2+i}}, \qquad 1\leqq i\leqq n/2,$$

$$\frac{dz_i}{dt}=-\frac{\partial H}{\partial z_{i-n/2}}, \qquad n/2<i\leqq n.$$

In this case the function $M = -H$, and is well known to be an "integral" of the system of equations. It follows that

$$\int \sum_{i=1}^{n/2} z_{n/2+i}\delta z_i - H\delta t$$

is a non-synchronized invariant. It is the invariant given by Cartan.

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